Pushing the Limits

Infinitesimal calculus

Everything is mathematical

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Antonio J. Durán

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Contents

Preface 7

Chapter 1. What is Infinitesimal Calculus and What is it Used for? 9
Functions 10
Derivatives 12
Integrals 17
The fundamental theorem of calculus 19

Chapter 2. From Archimedes to the 17th Century: the Foundations 23
Infinity in the Greek world 23
Archimedes 25
From Archimedes to the 17th century 28
Scientific features of 17th-century Europe 31
The calculation of quadratures and cubitures 35
Centres of gravity 39
The calculation of tangents 39

Chapter 3. Newton, the Last of the Magicians 43
The 'hero of reason' 45
The complicated life of a genius 47
Wooing science: *Principia* 52
Newton and infinitesimal calculus 56
A haughty genius 67
London: maturity and old age in the Treasury 68
Newton and his friends 69
The burial of Newton 73

Chapter 4. Leibniz, Master of all Trades 75
Leibniz and infinitesimal calculus 79
In the service of the Hanoverians 92
Leibniz's philosophy 95
The burial of Leibniz 97

Chapter 5. The Dispute over Priority 99
Initial mutual acknowledgements... although lacking sincerity 100
"Modesty is a virtue, but diffidence is a fault" 103
"You can tell the lion by its claw" 104
Fatio attacks... and Leibniz counter-attacks 105
The appearance of "Newton's ape" 109
Leibniz's fate was in the hands of the Royal Society 111
Charta Volans and the "leading mathematician" 114
The lion sharpens its claws 116
When Leibniz's patron became Newton's king 118

Chapter 6. The Infinitesimal Tamed 121
Newton, Leibniz and infinitesimals 122
"The ghosts of departed quantities" 124
Euler and the analysis of infinities 125
D'Alembert, Lagrange and Karl Marx 127
Cauchy 129
Euler, Cauchy and the aesthetic value of mathematics 134
Karl Weierstrass 135
Coda 137

Appendix. Euler and Infinitesimals 139

Bibliography 141

Index 142

Preface

Infinitesimal calculus is without a shadow of doubt the most powerful and effective tool for the study of nature that has ever been developed by mathematicians. The process for the birth and development of infinitesimal calculus spans a huge period of time, starting when mathematics became pregnant (if the metaphor holds) in the 3rd century BC, when Archimedes used the first infinitesimal arguments for the calculation of areas, all the way through to when Newton and Leibniz became its midwives, almost two thousand years later. To these greats we must add another century and a half during which Cauchy and Weierstrass 'tamed' infinitesimals by finding a suitable logical explanation for them.

Putting its seemingly mystical origins to one side (a sort of black magic based on the prodigious power of the infinite), infinitesimal calculus as we understand it today was made up of two apparently separate ideas – differential calculus, the fundamental concept of which is the derivative, and integral calculus. To these we must also add the bridge that links them, the fundamental theorem of calculus, which establishes that derivation and integration are inverse operations.

What makes infinitesimal calculus so versatile is the wide range of processes from mathematics, physics, technology, economics, and a wide range of other fields, that correspond to the calculation of a derivative, an integral or are intimately related to one of them, or both.

As an example, the derivative is also a fundamental concept in physics, since it corresponds to concepts such as instantaneous speed and acceleration, and thus, to force. It is not strange that the majority of the laws of physics are expressed in terms of differential equations, where functions and their derivatives are mixed. Consider another of the many possible examples of the versatility of calculus: the medical marvel that is an MRI scan. The behaviour of the X-ray wave that crosses our body during the scan is an integral, the value of which is the difference in the intensity between the wave that enters our body and the one that leaves it. The MRI machine then fills in internal anatomy based on the values of all these integrals.

In fact, modern physics was born with Newton, one of the inventors of infinitesimal calculus (among other things), and this is no coincidence: according to his own account, the ideas that would culminate in the discovery of *his* method of calculus were born mixed with his first reflections on gravity, and his initial, rudimentary version of infinitesimal calculus must have allowed him to make one of

his great discoveries: the deduction, using Kepler's laws of planetary motion, of the inverse square law governing the attraction of gravity.

A similar phenomenon took place with the other version of infinitesimal calculus, that of Leibniz: shortly after they were published in 1684 and 1686, the two seminal articles in which Leibniz set out *his* method of calculus had already been used to solve a wide range of problems from mechanics (the catenary, the brachistochrone, etc.), which had been unsolvable until that point, even for geniuses such as Leonardo da Vinci and Galileo.

Let's take a look at infinitesimal calculus and its history.

Chapter 1

What is Infinitesimal Calculus and What is it Used for?

Infinitesimal calculus is a scientific and technological tool of great importance. To properly understand the subtleties and complexities of this tool, we can begin by considering some of the problems that calculus helps us to solve. This also makes it possible to better understand its significance and why it is almost ubiquitous in modern science and technology.

There are many different applications. There are problems from physics, such as determining the velocity of a body given the distance it covers, or vice versa, given the speed at which it moves, to calculate the distance it covers. Other examples include calculating the speed at which a car moves given the power of its engine, or determining the position of a guitar string once it has been plucked.

There are also geometric problems, such as calculating the tangent to a curve, the area it encloses or its length. Many others appear in the space between physics and engineering, such as how to calculate a body's centre of gravity (of fundamental importance in naval architecture), calculating the position of a cable that is suspended between two posts (essential in hanging overhead electricity lines), calculating the temperature distribution on a metal plate to which a heat source is applied, and determining the movement of fluids (essential in designing fast jet aircraft). These are just a few examples and we could continue adding problems solved by calculus, making it practically infinite.

Indeed, the infinite is precisely the name by which the scientific and technological tool of infinitesimal calculus was known in the beginning. It is no coincidence that it is the infinite that provides infinitesimal calculus with almost magical power and effectiveness. The infinite lies at the heart of calculus, although in order to understand to what extent it penetrates it, we must first give some consideration to the fundamental concepts of calculus.

As we hinted in the preface, infinitesimal calculus is made up of two apparently separate arenas – differential calculus and integral calculus – each with their corresponding concepts and tools. The realm of differential calculus covers problems

such as the calculation of curves, or speed given the distance covered. Integral calculus covers the calculation of areas and volumes, or the distance travelled by a body, given its speed. To arrive at the starting point for both types we must first take on board the concept of a function.

Functions

A large proportion of the processes we humans wish to study, be they natural, economic or of any other source borne of our complex societies, can be modelled mathematically using functions. Put another way, functions are the language used by scientists for the mathematical study of all these processes.

A function is a rule that associates one number with another. This rule is commonly, although not always, expressed by means of algebraic or analytical operations on the corresponding number.

A simple example of a function would be the association of the following value with the number t:

$$\frac{t^2+1}{t^4+5}$$

Given that t will be assigned different numerical values, we call it a 'variable'. It is conventional to use the letters f, g, h, s and t to represent functions and the letters x, y and t for variables. Hence we write $f(t)$ for the number assigned to the generic number t by the function. In the previous example, we have:

$$f(t)=\frac{t^2+1}{t^4+5}$$

When we give the variable t specific values, we obtain the explicit values of the function; thus for $t=1$ we have:

$$f(1)=\frac{1^2+1}{1^4+5}=\frac{2}{6},$$

while for $t=2$ we have:

$$f(2)=\frac{2^2+1}{2^4+5}=\frac{5}{21}.$$

The following table shows the values of the function for a few other values of t.

t	$f(t)$
−1	2 6
0	1 5
√2	3 9

In physics, the simplest system is that of a body in motion. It can be described using the function s, which, for each moment in time t assigns the distance covered $s(t)$ by the body, or the function v, which for each moment in time t assigns the **speed, or velocity, $v(t)$ at which it is moving.**

Let us consider a concrete example: if after t seconds a body has moved exactly the square root of t metres, the function that defines the distance covered in this movement will be $s(t) = \sqrt{t}$. This function, which gives the distance covered by the body also conceals information about the velocity at which it moves, although we need to make use of differential calculus to reveal the mystery.

Let us consider another example: we have a body that moves for t seconds at a speed of t squared metres per second, the function which gives the velocity for this motion will thus be $v(t) = t$. Here we see something similar to the previous problem: the function gives the speed at which the body travels and also conceals information about the distance this body has travelled, although in this case to reveal the **mystery we need to make use of integral calculus.**

Similarly, functions can be used to model the most diverse problems: the behaviour of the share price of a given bank or company on the London Stock Exchange, the density of each small section of the human body (making it possible to examine bones, muscles and organs without needing to operate), or the pressure of the wind **against the wings of a plane in the air.**

Hence the first step in being able to apply the tools of infinitesimal calculus in order to solve a problem is being able to describe it in terms of functions.

Once the natural, physical and economic processes we wish to study have been reduced to functions, the fundamental concepts of infinitesimal calculus can then be used to extract the secrets they hold.

Derivatives

The fundamental concept of differential calculus is the derivative, and in fact it is not just one of the fundamental concepts of mathematics but of science in general. This is so because concealed beneath the concept are other basic ones such as speed and force (in physics), or the tangent to a curve (in geometry), to name just a few.

The derivative of a function f at a value a measures how the function varies at that value with respect to how the variable changes. Consider the two physics functions from the previous examples: $s(t) = \sqrt{t}$ and $v(t) = t^2$. Both assign the value 1 to $t = 1$: $s(1) = 1$ and $v(1) = 1$. However a table of values shows that the way in which both functions vary around $t = 1$ is quite different:

t	$s(t)$	$v(t)$
0.8	0.8944	0.64
0.9	0.9486	0.81
1	1	1
1.1	1.0488	1.21
1.2	1.0954	1.44

It can be observed that around 1 the function v varies more sharply than the function s.

To measure this variation, or rather to define the derivative, we take the general number a and a number close to it $a + h$, and use a quotient to compare the change between the values of the function at these numbers: $f(a + h) - f(a)$, and the difference between the values, $a + h - a = h$. The quotient is thus:

$$\frac{f(a+h)-f(a)}{h}.$$

Returning to our example of the functions $s(t) = \sqrt{t}$ and $v(t) = t^2$, we can see by how much these quotients vary for $a = 1$:

h	$\frac{s(1+h)-s(1)}{h}$	$\frac{v(1+h)-v(1)}{h}$
–0.01	0.5012...	1.99
–0.001	0.5001...	1.999
0.001	0.4998...	2.001
0.01	0.4987...	2.01

The largest value of these quotients for the function v approaches 2, whereas for the function s it is around 0.5; this clearly shows what was already apparent in the previous table: Around the value 1, the function v varies more sharply than the function s.

However, the quotient

$$\frac{f(a+h)-f(a)}{h}$$

is of interest to us precisely when $h = 0$, or rather when the number $a + h$ is the same as a.This value is referred to as the derivative of f at a and, using the notation of the French mathematician Louis Lagrange (1736–1813) (see Chapter 6), it is written as $f'(a)$. As we have been able to verify; the value of this quotient is $\frac{0}{0}$, in other words it is meaningless.

But this lack of meaning is only apparent, as the previous table shows, for our functions $s(t) = \sqrt{t}$ and $v(t) = t^2$, when h is small but not zero Both quotients,

$$\frac{s(1+h)-s(1)}{h} \text{ and } \frac{v(1+h)-v(1)}{h},$$

are clearly meaningful and suggest two numbers: 0.5 for the function $s(t) = \sqrt{t}$ and 2 for the function $v(t) = t^2$. Further on, we will see that these values in fact correspond to the derivatives of both functions at 1, or rather, $s'(1) = 0.5$ and $v'(1) = 2$.

This zero-quotient which is obtained in the definition of a derivative was, however, the difficulty that scientists in the 17th century (and those before them) came up

against every time they wished to calculate, for instance, the tangent of a curve or the instantaneous speed of a body in motion given the distance travelled by it.

The infinite, the essence of infinitesimal calculus, is camouflaged perfectly in this quotient of zero divided by zero. As we have just mentioned, the quotient

$$\frac{f(a+h)-f(a)}{h}$$

is of interest to us precisely when $h=0$, which just happens to be when the numerator and denominator are zero. Such quantities, which are zero but for which we nonetheless need to know the proportion between them, were christened by 17th century mathematicians as 'infinitesimals' or 'infinitesimal quantities' (if we permit ourselves the anachronism of using the concept of a function, which was implicit but not explicit in the mathematics of the time, to refer to infinitesimals).

The primitive infinitesimal calculus, developed by Newton and Leibniz and subsequently polished by Leonhard Euler (1707–1783) and other mathematicians in the 18th century, can be described as the art of handling infinitesimals. As we shall see in the following chapters, the paradox was that none of these mathematical geniuses was able to define with any logical rigour just what the infinitesimals upon which calculus was based were.

What Newton and Leibniz did manage however, was to bring to its conclusion work undertaken by many of their colleagues during the 17th century, by discovering infinitesimal calculus, one side of which is differential calculus. Specifically, what Newton and Leibniz achieved was to provide a few simple rules for use in resolving the meaninglessness of the division of 0 by 0 which they came up against every time they wanted to calculate the derivative of a function. These rules involved the explicit calculation of the derivatives of the most elementary functions, such as powers of the variable

$$(x^n)' = nx^{n-1},$$

trigonometric functions:

$$(\sin x)' = \cos x,\ (\cos x)' = -\sin x;$$

logarithms:

$$(\log x)' = \frac{1}{x},$$

exponentials:

$$(e^x)' = e^x, \text{ etc.,}$$

as well as rules for deriving the basic operations which can be carried out on functions, such as addition:

$$(f+g)' = f'+g',$$

multiplication:

$$(fg)' = f'g + fg';$$

division:

$$\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2};$$

composition:

$$(f(g))' = f'(g)g'.$$

The Gordian knot of infinitesimal calculus, as used during the 17th and 18th centuries, as well as at the start of the 19th, consists of defining precisely how to understand the value of the quotient

$$\frac{f(a+h)-f(a)}{h}$$

when $h = 0$. The knot was cut by the French mathematician Augustin Louis Cauchy (1789–1857) with a sharp sword made from the concept of the limit, which he himself defined in somewhat imprecise terms and which was subsequently refined by Karl Weierstrass (1815–1897), as we shall see in Chapter 6.

Given that the instantaneous speed at which a body moves is a derivative, it was also this difficulty of explaining the lack of meaning $\frac{0}{0}$ which paralysed physics until Newton freed it with the invention of infinitesimal calculus. The reader should bear in mind that prior to the invention of infinitesimal calculus at the end of the 17th century, it had only been possible to study the simplest of movements: uniform motion, when the distance travelled is proportional to the time, so the velocity is constant; and motion with uniform acceleration, when the distance travelled is proportional to the time squared and thus the speed is proportional to time and the acceleration has a constant value. This final type of motion, which governs, for example, the falling of a body due to the effects of gravity, would require all the scientific talents of a genius such as Galileo, who succeeded in understanding it decades

before the invention of infinitesimal calculus would turn the study of such motion into a school activity.

Let us illustrate all this with an example. As before, let us imagine a body in motion that covers a distance $s(t) = \sqrt{t}$ in time t (to make things more tangible, imagine that the time is measured in seconds and the distance in metres). The calculation of the average speed at which the body moves is easy. For example in the time interval from 1 to 4 seconds, the average speed will be the quotient of the distance travelled and the time taken:

$$\text{Average speed} = \frac{s(4)-s(1)}{4-1} = \frac{2-1}{3} = \frac{1}{3}\cdot\frac{m}{s}.$$

However, what happens if we wish to measure not just the average speed at a given time interval, but the instantaneous speed at which a body is moving at a given instant? To simplify the task, assume that we wish to calculate this instantaneous speed immediately after the first second once the body has started to move. To do so, let us take a time increment h and calculate the average speed between 1 and $1+h$:

$$\text{Average speed} = \frac{s(1+h)-s(1)}{1+h-1} = \frac{\sqrt{1+h}-1}{h}$$

To calculate the instantaneous speed at the first second, it suffices to reduce the time increment h to zero. But this results in the same thing happening as before and we have the meaningless formula:

$$\text{Instantaneous speed at one moment 1} = \frac{\sqrt{1}-1}{0} = \frac{0}{0}$$

This occurs because the instantaneous speed we are calculating corresponds to the value of the derivative of the functions which measures the distance $s(t) = \sqrt{t}$ at $t = 1$.

The previous numerical table shows that the value of this derivative should be 0.5. We shall now see that this is in fact its value, since the apparent meaninglessness of the division of zero by zero can be unravelled as follows:

$$\text{Average speed} = \frac{\sqrt{1+h}-1}{h}$$

Let us multiply the numerator and denominator by $\sqrt{1+h}+1$, and simplify

$$\text{Average speed} = \frac{\sqrt{1+h}-1}{h} = \frac{\left(\sqrt{1+h}-1\right)\cdot\left(\sqrt{1+h}+1\right)}{h\cdot\left(\sqrt{1+h}+1\right)} =$$

$$= \frac{1+h-1}{h\cdot\left(\sqrt{1+h}+1\right)} = \frac{h}{h\cdot\left(\sqrt{1+h}+1\right)} = \frac{1}{\sqrt{1+h}+1}$$

If in this final expression we reduce the time increment h to zero, we no longer have the same problem as at the start, and letting $h=0$ there is no longer a division by zero and, as we had suspected from the table, the value of the quotient at $h=0$ is precisely 0.5; in terms of physics, this means that

$$\text{Instantaneous speed at one moment } 1 = \frac{1}{2} = 0.5.$$

Thus we have unravelled the initial meaninglessness of the division of zero by zero in order to discover that if in t seconds a body moves $\sqrt{t}$ metres, after 1 second, it will be moving at a speed of 1/2 m/s.

Integrals

The other basic concept of infinitesimal calculus is the integral. Essentially, the integral measures the area generated by a function. Thus if we have a function f defined for values between a and b, the integral measures $\int_a^b f(t)dt$ in the following area:

The $\int$ symbol for representing the integral was invented by Leibniz (we shall return to this in more detail in Chapter 4), and is a stylised letter 'S', the first letter of the word 'sum'.

The integral is a much more versatile concept than that of mere area. In mathematics it can also be used to calculate volumes, lengths or centres of gravity, to name but a few examples. In physics, it corresponds to work – the work required to displace a body submitted to a force f between two positions a and b, and is expressed by:

$$\int f(t)\,dt.$$

The integral is also the tool required to calculate the distance travelled by a body given its speed. Let us now return to one of the physics problems with which we began the chapter. Four seconds after beginning to move, how far has a moving body travelled if at t seconds it moves at a speed of t squared a second? The answer is given by:

$$\int_0^4 t^2\,dt.$$

The problem we must now solve is how to calculate this integral. According to our interpretation of an integral as an area, it is the area enclosed by a section of the parabola. This area was already calculated by Archimedes two thousand three hundred years ago. It was one of the results that earned him his everlasting fame and which allows us to regard him as one of the most eminent forerunners of integral calculus. (We shall discuss Archimedes in greater detail in Chapter 2.)

The precise definition of an integral, without making recourse to the geometric concept of area, is a delicate matter in terms of logical rigour. Here, infinitesimals also come into play, although in a different way: in the figure on the previous page, we can see that the area is made up of vertical segments of length $f(t)$, where the number t takes values between a and b. In a certain sense, the drawing suggests that the area is the sum of these segments. However, since they are sections of a straight line, their area is 0, and as such it would seem that their sum is unable to generate an area. Once again we find ourselves with an infinitesimal quantity, the length of these segments, which must now be added up. In the notation invented by Leibniz

for the integral, the conception of the area delimited by a curve as a sum of infinitesimals is given symbolically: according to the diagram on the previous page, each segment of those which makes up the area has a height of $f(t)$ and, according to Leibniz, an infinitesimal width which is written as dt; the area of these segments will be its base multiplied by its height, or rather, $f(t)\,dt$, and the total area we wish to calculate is given by the sum: $\int f(t)\,dt$.

The meaning of this sum is something that neither Leibniz nor Newton, the fathers of infinitesimal calculus, would ever be able to explain. In fact, it was necessary to wait almost one and a half centuries for the first precise formulation of what is understood as an integral today. This feat is attributed to Cauchy and is also based on the concept of the limit (more details are provided in Chapter 6).

The calculation of areas – if the surface is not bounded by straight lines – is a highly complex task, and Newton and Leibniz's predecessors knew this well, in a certain sense, it corresponds to the calculation of an integral using their definition. It is the fundamental theorem of calculus which simplifies the calculation, although this is not always the case.

The fundamental theorem of calculus

Infinitesimal calculus is completed by the bridge between derivatives and integrals: the fundamental theorem of calculus which establishes that derivatives and integrals are inverse processes. More specifically, if we wish to calculate the integral

$$\int_a^b f(t)\,dt,$$

the fundamental theorem of calculus states that it suffices to calculate a function F such that

$$F'(t) = f(t)$$

for every t between a and b; hence

$$\int_a^b f(t)\,dt = F(b) - F(a).$$

The function f must also satisfy another condition, that of continuity, although for our purposes it is not necessary to enter into details.

Let us consider an example: the fundamental theorem of calculus makes it easy to calculate the following:

$$\int t^2\,dt\,.$$

This integral is highly versatile since, depending on how it is interpreted, it can be used to calculate the area bounded by a parabola, the area bounded by an Archimedian spiral, as well as the distance travelled by a body that moves with a speed $v(t) = t^2$. Making use of the fundamental theorem of calculus, it suffices to calculate a function with the derivative t^2. This can be done easily by using the rule for the division of powers: if

$$f(t) = t^n,$$

then

$$f'(t) = nt^{n-1}.$$

Hence it is easy to deduce that the derivative of the function $\frac{t^3}{3}$ is precisely t^2. Therefore:

$$\int_a^b t^2\,dt = \frac{b^3}{3} - \frac{a^3}{3}.$$

Returning to our example, the distance covered in 4 seconds by a body moving for t at a speed of t^2 seconds is given by the integral:

$$\int_0^4 t^2\,dt.$$

As such it suffices to calculate the previous formula with $a = 0$ and $b = 4$ to give:

$$\text{Distance covered} = \int_0^4 t^2\,dt = \frac{64}{3}.$$

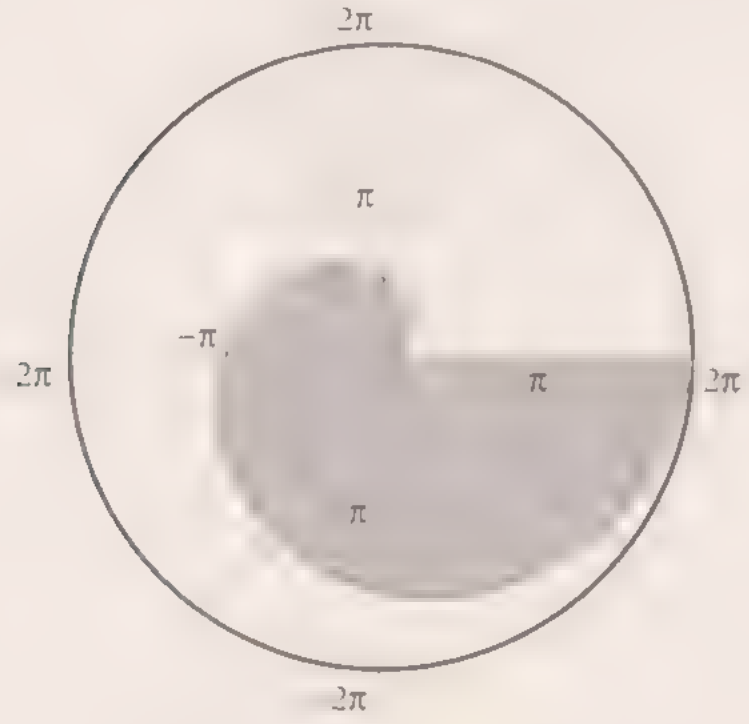
Let us now consider an Archimedean spiral, or rather the curve generated by a point that starts at the end of a line and moves uniformly along the line which also turns uniformly along its length. To make things clearer, imagine a point which moves at 1 m/s along a path which also turns at a constant speed. If we wish to calculate the area bounded by the first turn of an Archimedean spiral, we need to calculate the integral

$$\frac{1}{2}\int_{a}^{2\pi} t^2\,dt.$$

It suffices to calculate the previous formula with $a = 0$ and $b = 2\pi$ to give:

$$\frac{1}{2}\int_{0}^{2\pi} t^2\,dt = \frac{4\pi^3}{3}.$$

This was precisely the result obtained by Archimedes, although he expressed it in another way. "The area of the first turn of the spiral is equal to one third of the area of the circle whose radius is the length covered by the point on the line during this first turn." In fact, just as during the first turn of the spiral, the point will have travelled a distance of 2π along the straight line, the circle covering this radius will have an area of $\pi (2\pi)^2 = 4\pi^3$, hence Archimedes' formulation.



It is beyond the scope of a book such as this to provide more detail of these brief sketches of the concepts and methods of infinitesimal calculus. Equally as fascinating, or perhaps even more so, is the way in which mathematicians discovered and developed them. In the following chapters, we will narrate this intellectual epic which lasted almost two thousand years, and the reader will be able to see how the hands of Newton, Leibniz, Euler, Cauchy and other geniuses of mathematical thinking modelled the concepts of differentials, derivatives, integrals and limits until they took the shape in which we know them today and the way in which we use them.

Chapter 2

From Archimedes to the 17th Century: the Foundations

Throughout infinitesimal calculus' entire incubation period – almost two millennia from Archimedes in the 3rd century B.C. to Newton and Leibniz in the 17th century – a large number of diverse mathematical ideas and theories would be accumulated and established. The rebirth of Greek knowledge, in particular the work of Archimedes, the availability of a more sophisticated numbering system than that of the Greeks or Romans, and above all, the development of symbolic algebra and analytic geometry, making it possible to represent curves algebraically and allowing the algorithmic treatment of problems such as tangents, quadratures, centres of gravity, maxima and minima, etc. In short, symbolic algebra and analytic geometry constituted a language suitable for describing the initial moments of the life of calculus. Its birth also required contributions from many scientists, largely made during the 17th century and giving rise to a vast range of results.

The full process is one of great complexity, intensity and passion, not only from a scientific point of view, but also, and perhaps even more so, from a historical one. It was affected by the great historical events of humanity, symbolised in the loss and recovery of classical Greek culture, the development of the scientific revolution, and, to give another example, the problems of isolation derived from the political situation and the wars that took place in 17th century Europe. However, it was also influenced by other, lesser, human miseries, such as some scientists plotting against others, fights, slander and disputes.

Infinity in the Greek world

Let us begin at the beginning, or rather with classical Greece. At that time, mathematics and philosophy were making their first forays into the realm of the infinite, where the metaphysical reflection of calculus resides.

For the Greeks, the infinite became a two-headed monster: the infinitely small and the infinitely large. The idea of the infinite soon became embroiled in scandals and disputes. It would appear disguised in the impossibility of measuring the diagonal of a square, a fatal event for the Pythagorean world view which triggered the first crisis in the foundations of mathematics. It was also present in Zeno's paradoxes of plurality and motion, which came to show, among other things, the dialectic conflict between various schools of Greek philosophy and also served to illustrate the influence these conflicts had on mathematics.

Under these circumstances, the prohibition, or rather the regulation, of the use of the infinite was not long in coming. Hence, faced with the impossibility of denying infinite processes, Anaxagoras would write that there is no "least of what is small, but there is always a smaller," just as "there is also always something greater than what is great." Aristotle attempted to control them by prohibiting actual infinity: "The infinite cannot be an actual thing and a substance and principle," he writes in book III of his *Physics*, although he himself recognises further on that "to suppose that the infinite does not exist in any way leads obviously to many impossible consequences," such that "the alternative then remains that the infinite has a potential existence," or rather that "everything that is infinite may be so in respect of addition or division." In other words, "magnitude is not actually infinite. But by division it is infinite." By means of example, this Aristotelian control makes it impossible to conceive of a segment as a collection of infinite points on a line yet does make it possible to divide the line in half, for example, as many times as we wish.

In terms of the role of the infinite in mathematics, Aristotle writes: "Our account does not rob the mathematicians of their science, by disproving the actual existence of the infinite in the direction of increase, in the sense of the untraversable. In point of fact they do not need the infinite and do not use it. They postulate only that the finite straight line may be produced as far as they wish."

However, from a mathematical point of view, another preceding phrase is of greater importance: "Every finite magnitude is exhausted by means of any determinate quantity however small." This is known as 'Archimedes' Axiom', or 'the continuity axiom'. In fact it was first stated and used by Eudoxus, a follower of Plato, one of Aristotle's contemporaries. This principle and the definition of equal ratios allowed Eudoxus to overcome the crises created by the discovery of incommensurable numbers. Archimedes' axiom later appears in Euclid's *Elements of Geometry* as a definition: "If a first magnitude has to a second the same ratio as a third to a fourth, any equimultiples whatever of the first and third will also have the same ratio to any

equimultiples whatever of the second and fourth respectively, taken in corresponding order." Based on this axiom, Eudoxus created the so-called 'method of exhaustion', a rigorous procedure for proving the calculation of areas and volumes which would be used to show, among other things, that two circles are to each other as the squares of their diameters – the common proportion that is today referred to as *pi*. The method and the result were subsequently collected by Euclid in his *Elements*.

Archimedes

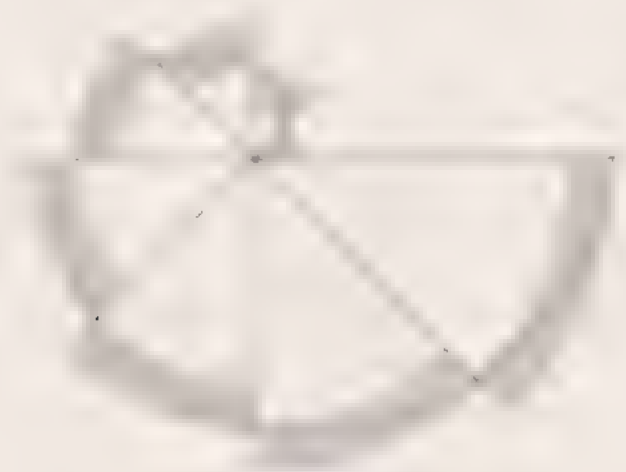
However, without a doubt, the figure who knew how to make masterly use of the method of exhaustion was Archimedes. In a number of his works, he states the axiom of continuity: "The excess by which the greater of two unequal areas exceeds the less can, by being added to itself, be made to exceed any given finite area," he writes in *On the Quadrature of the Parabola*. However, he recognised that he was not the first to use it: "The earlier geometers have also used this lemma," he wrote in reference to Eudoxus.

Of Archimedes' many applications of the method of exhaustion, we shall focus on its use to calculate the area of the spiral. Archimedes studied an equiangular spiral, or rather one in which, and according to the definition in the previous chapter, the

THE ARCHIMEDES PALIMPSEST

In 1906 in Constantinople, the Danish scholar Johan Ludvig Heiberg found a palimpsest – an ancient manuscript which conserves traces of a text which had previously been artificially erased – with works by Archimedes. In this instance, the mathematical content had been concealed by the creation of a prayerbook for Sunday liturgies and other Christian festivals. The works that were discovered included one which was until then unknown: *The Method*. The calligraphy indicated the manuscript may have been written around 975 A.D., while the religious writings that cover over Archimedes' works date to approximately 1229.

curve generated by a point that starts at the end of a ray and moves uniformly along the ray which also turns uniformly along its length. Archimedes claimed that the section of the spiral generated upon the ray completing a full turn encloses an area equal to one third of the area of the circle whose radius is the length covered by the point on the line during this first turn. In order to prove this, he created a circumscribed approximation of the area of the spiral by means of a fixed number, let us say n, circular sections – obtained by dividing the circle into n equal parts – contained in the spiral and another inscribed approximation, also with n circular sectors but containing the spiral, as shown in the figure below:



In fact, these approximations for calculating the area of the spiral are precisely the same as those which are today used to calculate, by means of an integral, the area of a given curve in polar coordinates, and are equivalent to the rectangular approximations commonly used to define the definite integral of a function in a given interval.

This is one of the reasons why we must go all the way back to Archimedes if we wish to find the first records of integral calculus.

However there is also another important reason that allows us to bestow upon Archimedes the title of the first forefather of integral calculus. However, regrettably this reason did not influence the history of subsequent mathematics. We are referring to the contributions contained in the lost book, *The Method*.

In Archimedes' heuristic arguments which are contained in *The Method*, we find another record of integral calculus, an idea which would not reappear in mathematics until two thousand years after Archimedes in the 17th century. Furthermore, the idea went against the Aristotelian prohibition of actual infinity.

This revolutionary idea was to consider an area as a collection of lines (for example, a rectangle would be the collection of lines parallel to one of its sides), or a volume as collection of areas (a cylinder would thus be the collection of circles parallel to its base). By necessity, these collections would tend to be infinite, so the actual infinite, which had been prohibited by Aristotle, appears.

THE VALUE OF INFINITY

infinite terms. The sum in question is as follows:

$$1+\frac{1}{4}+\frac{1}{4^2}+\frac{1}{4^3}+\cdots$$

the form $1-\frac{1}{4}$. This gives:

$$\left(1+\frac{1}{4}+\frac{1}{4^2}+\frac{1}{4^3}+\cdots\right)\left(1-\frac{1}{4}\right)=1+\left(\frac{1}{4}-\frac{1}{4}\right)+\left(\frac{1}{4^2}-\frac{1}{4^2}\right)+\left(\frac{1}{4^3}-\frac{1}{4^3}\right)+\cdots$$

And since $1-\frac{1}{4}=\frac{3}{4}$, dividing gives us:

$$1+\frac{1}{4}+\frac{1}{4^2}+\frac{1}{4^3}+\cdots=\frac{4}{3}$$

one, the sum of which is finite.

As we have already mentioned, this idea did not resurface in mathematics until the 17th century in the work of figures such as Bonaventura Cavalieri and Grégoire de Saint Vincent, as we shall see further on. These mathematicians were familiar with the work of Archimedes: his books were in print towards the second half of

the 16th century), but not *The Method*, and so they were forced to rediscover Archimedes' technique and that work would play an essential role in the birth of calculus.

According to chroniclers, Archimedes died at the hands of a soldier during the Roman conquest of Syracuse in 212 B.C. Judging by this mosaic, which was recovered from Pompeii, the blood of the sage was spilt as he attempted to safeguard his records

From Archimedes to the 17th century

Prior to the assimilation of the works of Archimedes in the 17th century, thus accelerating the arrival of infinitesimal calculus, it seems appropriate to discuss some of the medieval and Renaissance contributions which were necessary for the birth of calculus. However the most important of these are not directly related to calculus, and as such this will only be a brief overview. Primarily, it concerns the complete process of the loss and subsequent recovery and assimilation of the Greek legacy. The diffusion of the Indian numbering system throughout the whole of Europe was also essential, a long and complex learning process that effectively began at the start of the 10th century until consolidated by the *abacus schools*, centres of learning which began to proliferate in Italy from the 13th to the 16th century.

At the end of this period, both rational and irrational numbers had begun to be expressed as decimals, albeit in a somewhat unwieldy notation. The key figure behind the idea, together with François Viète (1540–1603), was Simon Stevin (1548–

1620). Born in Bruges, Stevin developed decimals for mainly utilitarian reasons. "Decimal numbers are a class of arithmetic based on the idea of the base 10 progression, making use of Arabic numbers, such that each number can be written and any operation from the world of commerce can be carried out using just integers, without the help of fractions." Based on his decimal theory he proposed the unification of weights, measures and coins into a metric systems, an idea that would not become reality until the French Revolution.

Shortly after, other authors began to use the same idea as Stevin, but with the modern notation of the decimal point to indicate the start of the decimal portion. These included figures such as John Napier, one of the inventors of logarithms. Logarithms arose at the start of the 17th century and were closely related to the discovery of infinitesimal calculus. They were an independent creation of the Scottish noble John Napier (1559–1617) and the Swiss Joost Burgi (1552–1632). At first, logarithms were an aid to numerical calculation, more specifically for simplifying multiplications involving large numbers of digits that were common to astronomical calculations. It is easy to imagine the time spent, the complexity and risk of making mistakes inherent to a large number of these multiplications. In the words of Napier, one of the creators of logarithms, "there is nothing that is so troublesome to mathematical practice, than the multiplications, divisions, square and cubical extractions of great numbers, which besides the tedious expense of time are... subject to many slippery errors."

To simplify such multiplications, at that time there was a method referred to as *Prosthaphaeresis*, which was based on a trigonometric formula, that which transforms a product into a sum. In fact, John Napier created logarithms to simplify this method by means of the preparation of tables which directly transformed products into sums.

The method of *Prosthaphaeresis* proceeds as follows. Assume we wish to multiply two large numbers, n and m. Let us assume each has eight digits, something which was not uncommon in the astronomical observatories of the time. To do so, we must find two numbers a and b in a table of cosines such that $n = \cos(a)$ and $m = \cos(b)$; we must now look up the value of $\cos(a-b)$ and $\cos(a+b)$, in the table and finally make use of the following trigonometric formula:

$$\frac{\cos(a+b)+\cos(a-b)}{2} = \cos(a)\cdot\cos(b) = n\cdot m.$$

Had we proceeded directly to the multiplication, we would have had to multiply one of the numbers by each of the eight digits of the other and then add the eight resulting numbers, each with eight or nine digits. On the other hand, by using this formula, we have reduced the product through the use of the trigonometric tables to three additions and a simple division by 2.

The *Prosthaphaeresis* method was, in some senses, a technological tool (the time it saved made it the first ancestor of the calculator) and for a while it was kept secret, remaining in the hands of a privileged few. Napier, for example, became aware of it according to a somewhat far-fetched account, from John Craig, doctor to the king of Scotland and Napier's friend, who had participated in a journey to Denmark towards the end of the 16th century, in search of the future wife of the king. Storms forced their boat to seek refuge on the coast, in a place close to the best astronomy of the time, which Tycho Brahe had set up on the island of Ven, between Denmark and Sweden. The expedition party was received and it was there, during the long and interminable days and nights of the storm, where the doctor learned the *Prosthaphaeresis* method, which he taught to John Napier upon his return.

ORESME'S PROOF

In the words of Oresme himself, the reason why the sum of the harmonic series

$$1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\ldots$$

is infinity is as follows: "To a magnitude of 1 foot add $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$ foot, etc., the sum of which is infinite. Indeed, it is possible to form an infinite number of groups of terms with a sum greater than $\frac{1}{2}$.

Thus $\frac{1}{3}+\frac{1}{4}$ is greater than $\frac{1}{2}$, since each of these two terms is greater than $\frac{1}{4}$.

Similarly,

$$\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}$$

is greater than $\frac{1}{2}$, since each of these four terms is greater than $\frac{1}{8}$.

Similarly,

$$\frac{1}{9}+\frac{1}{10}+\ldots+\frac{1}{16}$$

is greater than $\frac{1}{2}$, since each of these eight terms is greater than $\frac{1}{16}$, and so on *ad infinitum*."

Prior to the 17th century, contributions which were directly related to infinitesimal calculus were rather scarce. However, those of the Parisian scholar Nicole Oresme (1323?–1382, are worth mentioning. They included a primitive seed of the idea of the function and its graphical representation. "Every measurable thing except numbers is imagined in the manner of continuous quantity." He also made contributions to the emerging study of infinite sums and series, proving for the first time in history that the sum

$$1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\cdots$$

was infinite.

Scientific features of 17th-century Europe

Before moving on to the advances taking place during the 17th century that would form the origins of infinitesimal calculus, we should first consider certain particular features of European science at the start of that century.

In the first place it should be noted that mathematics and science in general, were not professionalised like they are today, nor was scientific effort geared around universities and nor were there a large number of channels to disseminate results, in fact there was perhaps even a culture of not doing so. Hardly any of the figures who appear in the following pages were mathematicians by profession. Some were lawyers, others architects, there were also diplomats and theologians. But there were few, very few, who earned their living as mathematicians or who had links with the universities. (As such, when we use the word mathematician in this book, we use it to indicate that the figure in question made contributions of mathematical interest, despite being professionally or scientifically involved in other endeavours.)

This gave rise to a series of drawbacks: the research was structured in groups clustered around a certain scientist or figure with an interest in science, groups that were often isolated from each other and did not come into contact, be it for patriotic reasons or due to the disputes generated by the frequent scientific challenges and competitions of the time. All this resulted in the poor dissemination of any results that were obtained. It was common for them to be communicated in a letter to a friend or acquaintance. With time – sometimes a very long time – this knowledge was included in books, although even this was an unreliable way of sharing knowledge.

Under these circumstances the best way of ensuring a good mathematical education was not by going to university but by finding a private tutor. The main mathematical circles of the first half of the 17th century were in the *Academia dei Lincei* (the Lincean Academy), to which Galileo belonged, founded in Rome in 1603 but which had disappeared thirty years later. Perhaps the most important circle was grouped around the friar Marin Mersenne (1588–1648) who belonged to the Order of the Minims. Living in Paris from the second decade of the century, Mersenne formed a group of mathematicians and scientists who met on a weekly basis. However, this group also served as a link between many European philosophers and scientists who maintained regular correspondence with Desargues, Fermat, Pascal – who attended the meetings of the Mersenne circle at the end of the 1630s, being only an teenager at the time –, Descartes and Galileo (the circle also contributed to the dissemination of the philosophical and mathematical work of Descartes and the astronomical work of Galileo). Aside from his important coordinating role, Mersenne also made some of his own, more minor, contributions in the field of mathematics and acoustics.

We can say that at the start of the 17th century the mathematical and scientific legacy of the Greeks, which survived the cultural earthquake wrought by the Middle Ages, had almost been recovered in its entirety. However, although the scholars of the period had good knowledge of Euclid's *Elements* and other basic works, a few more decades would pass until more in depth and difficult texts were mastered, such as those of Archimedes. The assimilation of these latter texts was essential to the discovery of infinitesimal calculus, in fact some of the most important predecessors of calculus, such as Wallis and Barrow, possessed editions of the works of Archimedes. It suffices to note that Archimedes was the most commonly cited author throughout the century with respect to the calculation of areas and volumes.

However, there was one aspect of Archimedes' mathematics, as well as of Greek mathematics in general, which was radically different: logical rigour. The mathematics of the Enlightenment period were much less rigorous than those carried out by the ancient Greeks. This fact would at first appear to be a regression, although it was in fact this change of attitude that finally made it possible to overcome the boundaries of Greek mathematics, and in particular the discovery of calculus. Effectively, the mathematicians of the 17th century were interested in discovery above all else, not just in offering impeccable logical proofs of their findings, as was the case with the Greeks.

Why this change of attitude? We could suggest various reasons: the pressure of Platonic philosophy, which was largely responsible for the requirement of rigour present in Greek mathematics, did not exist in the 17th century. There were historical reasons too. The 16th and 17th centuries were full of great discoveries of all kinds, not least geographical ones. America was discovered towards the end of the 15th century, a product not just of logical rigour, but also of the error made by Columbus in his calculation of the Earth's radius. In science there was Copernicus' heliocentric theory, the double circulation system of the human body, and in technology Gutenberg's press, the microscope and the telescope.

In summary, mathematicians preferred to concentrate their efforts on the development of suitable methods for discovery, in spite of the fact they may have lacked rigour. Such circumstances were favourable to the reappearance of the infinite, free from Aristotelian limitations, allowing infinitesimals and infinitely large quantities to become powerful tools. First of all they were used to solve specific problems in the

THE INFINITE IS A MATTER FOR GOD

and contain as much reality as possible."

and a believer in an all-powerful God

calculation of areas, volumes, tangents, centres of gravity, maxima and minima, etc. The work was carried out by an eminent group of mathematicians, referred to as the forefathers of calculus, during the first early and mid 17th century. Later on, they became conceptual tools which made it possible to develop two similar but not equal versions of infinitesimal calculus, those of Newton and Leibniz. And finally, in the 18th century, they were used by Euler, without doubt the great genius in the handling of infinities, to found mathematical analysis as the study of functions by means of infinitesimal methods.

To complete this panoramic view of the circumstances which favoured the birth of calculus, we must now give consideration to the other large thread of 17th century mathematics: analytic geometry.

In short, analytic geometry made it possible to associate an equation with a curve. For example, a circumference of radius 1, or rather the curve defined by the property that all its points are one unit from a fixed point referred to as the centre, has the equation $x^2 + y^2 = 1$. Going in the other direction, it also made it possible to associate a curve with an equation, which immediately had the effect of increasing the number of curves studied by mathematicians, given that it was much easier to define new ones using equations instead of geometric properties, as had been the case until then. Furthermore, it also made possible the application of algebraic procedures to the study of geometric problems such as those involving areas, tangents, etc.

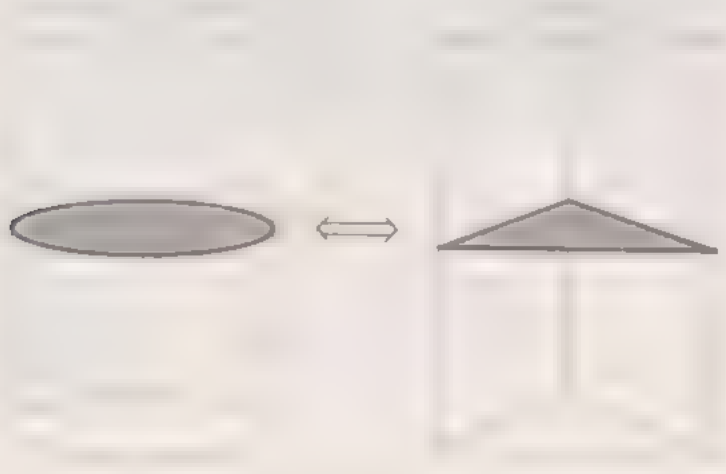
The resulting advantage is a consequence of using the more general procedures of algebra instead of the more specific ones of geometry. For example, turning history back to front, compare the specificity of the geometric calculation of the tangents of a curve (a procedure which differed radically for each curve) to the same calculation using an analytic process (the calculation of the derivative of a function), which is more or less the same for all cases and is carried out by applying the algorithmic process based on the rules of derivation.

Now returning to view history in the correct direction, it becomes possible to understand the importance of the unifying potential of the use of algebra in the realm of geometry, made possible by analytical geometry, in the discovery of these general rules, concealed behind an enormous quantity of specific cases accumulated in the first three-quarters of the 17th century, making it possible for Newton and Leibniz to discover calculus.

The calculation of quadratures and cubitures

Let us now move once and for all to the start of the 17th century to follow in detail the infinitesimal procedures that gave rise to calculus. We shall begin with those dealing with the calculation of areas and volumes, or to use the terminology of the time, **the calculation of quadratures and cubitures.**

Of the procedures that arose for resolving these calculations during the first third of the century, the most influential was that of indivisibles, attributed to the Jesuit Bonaventura Cavalieri (1598–1647), student of Galileo who taught in Bologna (perhaps here it is necessary to exclude the methods developed by Kepler for the calculation of the volume of solids of revolution, which helped the Austrian wine merchants to design their barrels and casks). Generally speaking, we can state that indivisibles represented the rediscovery of the methodological foundations of Archimedes' mechanical method. Cavalieri believed areas to be made up of segments and volumes of sections of flat areas; he gave the name *omnes linae* (all lines) to the set of segments made up by a flat area. For us, the interest of these *omnes linae* lies in the fact that we can compare the area of two flat shapes by comparing their corresponding *omnes linae*, according to Cavalieri. "The ratio between the area of two shapes is equal to the ratio between all the collections of lines, taken with respect to the same *regula*," as shown in the diagram.



Cavalieri extended his ideas not just to areas, but also to solids and attempted to develop a complete theory of indivisibles which, avoiding intimate processes, as had done the Greeks, allowed him to state his results. However, an analysis of his work clearly shows the presence of actual infinity in his arguments. And in a certain sense this was positive, since it was precisely the more explicit presence of infinity that made the Cavalieri method more flexible for discovery in comparison with the

method of exhaustion employed by the Greeks, despite it being less rigorous. In fact, by using his method of indivisibles, Cavalieri was able to calculate the areas of the general parabolas x^n, for $n = 3, 4, 5, 6$ and 9, going much further than Archimedes, whose calculations of the area of a parabola and a spiral were limited to x^2.

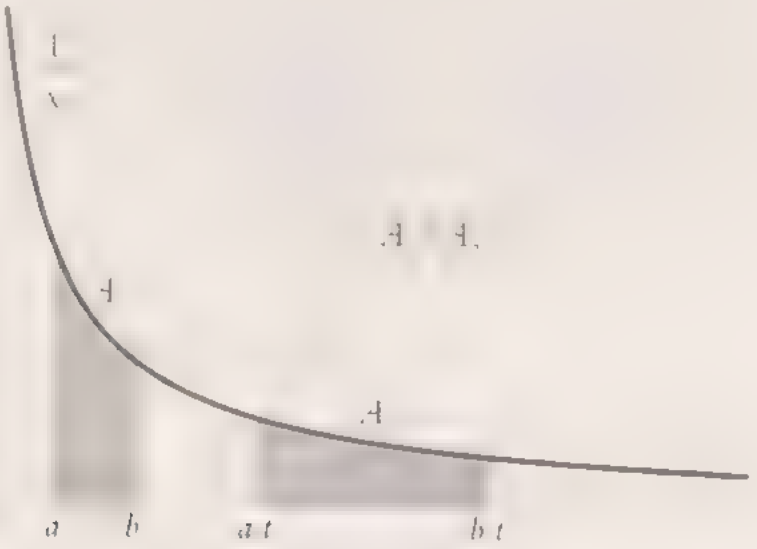
Compared with what was to come later, Cavalieri's method of indivisibles had a number of drawbacks: generalised use was difficult, it depended excessively on geometric arguments and procedures, and this is to say nothing of its weakness in terms of logic. However, from that point onwards, it opened up the discovery of new quadratures and cubatures, which made this aspect of mathematics more advanced than the achievements of the Greeks.

Moreover, the drawbacks of the method of indivisibles were soon overcome. Thus, Evangelista Torricelli (1608–1647), Cavalieri's friend, made masterly use of them while also providing rigorous Greek-style proofs, whereas Fermat, Pascal and Wallis – not to mention the infinitesimals of G P Roberval (1602–1675) – transformed Cavalieri's geometrical procedures into arithmetic ones, thus expanding their generality and potential for use.

A close-up of a marble statue of Bonaventura Cavalieri which is held in the Academy of Fine Arts of Brera, Milan. The Jesuit mathematician has been immortalised reflecting upon the calculation of indivisibility

Before moving on to consider Wallis's application of arithmetic to Cavalieri's method, we should first consider Gregoire de Saint Vincent (1584–1667), a Jesuit and follower of Christopher Clavius (whom Pope Gregory XIII charged with the reformation of the calendar and who was a fierce advocate of the study of

mathematics among the Jesuits) and, for a short while, tutor in the court of Philip IV of Spain. He made a number of important contributions. On the one hand, he developed a method for geometric integration which would later influence Pascal, although the delay in publication (it was most likely ready in the last years of the 1620s but remained unpublished until 1647), meant that interest in it was weakened due to the more algebraic methods developed under the influence of analytic geometry. On the other, he developed a study of the geometric series, which Huygens recommended Leibniz study, together with some applications, one of which includes a discussion Zeno's famous paradox of Achilles and the tortoise: Saint-Vincent argued that Zeno had failed to take into account that, in Achilles' chase, time formed a geometric progression with a ratio of a half and, as such, although the number of time intervals is infinite, their sum is finite. However, perhaps Saint Vincent's most outstanding contribution is the relationship that he discovered between the area bounded by a hyperbole and logarithms. Expressed in the parlance of the time, he showed that if the length of the intervals grows geometrically, the area that is bounded grows arithmetically, as shown below.



Let us now turn to John Wallis (1616–1703), one of the founders of the Royal Society of London, who held the Savilian chair of geometry at Oxford University from 1649 (his status was boosted by the fact that he had deciphered certain messages intercepted by the parliamentarians during the English civil war). Similarly to Barrow, he was responsible for a bilingual edition in Latin and Greek of the works of Greek authors including Archimedes, and also wrote a grammar of the English language (1653).

In terms of the issue that interests us, Wallis expurgated Cavalieri's indivisibles from his geometric package, assigning them numerical values instead. As such, he transformed geometric multiplications for the calculation of areas into arithmetic calculations with a primitive limit step. Wallis had already made shameless use of infinite processes (it should be noted that we owe the symbol currently used to represent infinity, ∞, to him), taking a step forward in an age blessed with more powerful methods for discovery over rigour. An excellent indication of these changes can be observed if we compare the titles of the works of Cavalieri and Wallis: the *Geometria Indivisibilibus Continuorum Nova Quadam Ratione Promota* of the latter, compared to the *Arithmetica Infinitorum* of the former. That is to say, the generality of arithmetic and algebraic calculations compared with the specificity of geometry, or infinity's power for discovery compared with the limitations imposed on the Greeks. Wallis' consideration of logical rigour is illustrative of the thought of the time: "This procedure is highly heterodox, but can be verified by the well-known method of inscribed and circumscribed shapes, which is not necessary, since frequent iteration causes nausea among readers. Anyone familiar with the material can carry out the proof." This is one of the few occasions on which the term 'proof' or 'demonstration' appears in Wallis' book. Allowing himself to be guided by his arithmetic method, incomplete induction and his intuition, he managed to discover the area of all generalised parabolas y, where r was any rational number, with the exception of -1. Moreover, as a consequence, he found a fascinating formula for the number π:

$$\frac{\pi}{4} = \frac{2\cdot 4\cdot 4\cdot 6\cdot 6\cdot 8\cdot 8\cdots}{1\cdot 3\cdot 3\cdot 5\cdot 5\cdot 7\cdot 7\cdots}$$

Wallis' arithmetical methods for calculating areas had an enormous impact on Newton, who said that the development of the binomial theorem and other initial ideas about calculus were rooted in the careful study he made of Wallis' book when he was a student at Cambridge. Wallis himself proposed a curious genealogy of infinitesimal calcus:

1. Method of exhaustion (Archimedes).
2. Method of indivisibles (Cavalieri).
3. Arithmetic of infinities (Wallis).
4. Method of infinite series (Newton).

Centres of gravity

Closely related to the calculation of areas and volumes was the determination of centres of gravity. The appearance of Archimedes' treatise *On the Equilibrium of Planes* meant that, at the end of the 16th century, a number of mathematicians were concentrating their efforts on solving these problems, including one of the translators of Archimedes into Latin, Francesco Maurolico (1494–1575) and Federico Commandino (1509–1575), as well as Simon Stevin, who simplified Archimedes' methods.

Further on, we see the works of Paul Guldin (1577–1643) from Switzerland, who rediscovered a theorem regarding the volumes of solids of revolution and centres of gravity that is now known as Guldin's theorem, although it had already been published in Pappus' *Mathematical Collection*: "The volume of a solid of revolution is equal to the area of the surface it generates multiplied by the circumference described by its centre of gravity". Guldin sustained a fierce polemic with Cavalieri, both of whom were Jesuits, with respect to the latter's method of indivisibles. On the one hand, he accused him of having plagiarised Kepler, and on the other, he cited the lack of logical consistency inherent in considering an area as a collection of segments. In fact, he provided a simple but nonetheless ingenious geometric construction in which a clear contradiction arose when using Cavalieri's method of indivisibles. It goes without saying that the way in which Guldin himself justified the theorem that bears his name, plagued with metaphysical arguments, is even more dubious than Cavalieri's methods, something which the latter, in retaliation for Guldin's attacks, wasted no time in pointing out.

The calculation of tangents

The infinitesimal methods related to the calculation of tangents constitute, together with those of areas and volumes, the main problems whose study finally crystallised into the birth of calculus.

The concept of the tangent itself, "a straight line that comes into contact with the curve at a single point", began to be problematic, since the ease of creating new curves resulting from Fermat and Descartes' analytic geometry resulted in a vast expansion in the number of curves considered and studied by mathematicians and scientists. Logarithms are an interesting example of this evolution: they began their life as a tool to facilitate the enormous quantity of multiplications, divisions and extractions of roots for very large numbers which were carried out in astronomical observatories as the tables that indicated the positions of the stars became increas-

ingly precise and ended up as a curve for which it was possible to calculate the area, tangent, etc. The increase in the number of curves that were studied, in addition to making it difficult to apply the old Greek definition of a straight line which touches the curve at a given point, also created a requirement to develop new methods for calculating the tangents of the new curves. These included Fermat and his method of *ad equalitas*, which was also vital for the study of maximum and minimum problems and in straightening curves. On account of these contributions, as well as his work on quadratures, certain 18th century mathematicians (fellow Frenchmen) considered him as the inventor of calculus. The claim is somewhat exaggerated, although Newton, himself, in a letter discovered in 1934, recognised that his ideas for the development of calculus were taken from Fermat's method for tangents: "I had the hint of this method from Fermat's way of drawing tangents and by applying it to abstract equations, directly and inversedly, I made it general." At any rate, Fermat, 'the prince of amateurs', as he is referred to by the Scottish writer and mathematician Eric Temple Bell, in reference to his non professional interest in mathematics, holds a place of honour in the discipline's history, and not only on account of the narrow margin in which he was unable to write the alleged **proof of his famous final theorem.**

However, in addition to Fermat, there were also a handful of other mathematicians who contributed new methods for the calculation of tangents, drawing on a range of ideas, practically all of which were based on the use and abuse of infinitesimal quantities. Examples include Roberval and his cinematic method for calculating the tangent of a spiral, also used by Galileo and Torricelli, as well as Archimedes in ancient Greece. Descartes and his circular method presented in *Geometry*, and also Barrow, Hudde and Sluse with their pseudo differential methods. All of these presented similar drawbacks: they worked reasonably well for algebraic curves, despite requiring specific adaptation depending on the curve in question (an adaptation of great complexity and even impossible to carry out for transcendental curves). All the methods would be united under Leibniz' concept of the differential and Newton's concept of fluxion and the rules for their calculation, that is to say something similar to our current derivative and the rules of derivation.

Related to the tangent problems, in the second half of the century, a different and historically important problem arose: it involved the inverse of tangents, or rather determining a curve given a property of its tangents. The first to formulate one of these problems was Florimond de Beaune (1601–1652), a jurist and follower of Descartes. Perhaps the most famous problem of those proposed by De

Beaune was the calculation of the curve with a constant subtangent. Descartes himself failed in his attempts to solve the problem, leaving the glory for Leibniz. As we shall see further on, its solution was included in the first publication in the history of infinitesimal calculus and permitted Leibniz to demonstrate the power of his method to the world.

One of the essential and unavoidable elements for being able to discover calculus was the recognition that tangents and quadratures are inverse problems. In current terminology, there is an inverse relationship between derivation and integration, a fact that we now justifiably refer to as the fundamental theorem of calculus. Fermat, Torricelli and above all, Barrow were already aware of this. However for reasons that we shall soon see, they did not understand the essential importance of this result, not only as a problem-solving tool, but as the driving force in being able to connect two types of problems, which were in principle different.

Isaac Barrow (1630-1677) is one of the giants in the famous phrase Newton sent to Robert Hooke (a hunchback) in a letter dated February 1676: "If I have seen further it is by standing on the shoulders of giants." Barrow was Newton's teacher at Cambridge and the first holder of the Lucasian chair, from which he resigned in 1669 (substituting Newton, in his place) to dedicate himself to religion. He was probably the mathematician who came closest to discovering calculus, if we

Isaac Barrow was Newton's teacher. His works form part of the foundations of infinitesimal calculus

exclude Newton and Leibniz. He developed a method for the calculation of tangents which was extremely close to our derivatives. He also made important contributions to the calculation of areas and, above all, provided an acceptable proof of the inverse relationship between the calculation of tangents and areas, a relationship which he probably began to perceive through the ideas of Torricelli, whom he met when exiled from Cromwell's England. The result and proof are to be found in "Lection X" of his *Lectiones geometricae*. It is exclusively geometric, valid for uniform curves, and uses the old Greek concept of a tangent as a straight line which touches a curve at just one point.

What was Barrow missing? Nothing other than going from the specific problem of calculating tangents to the more general one of calculating the variation of a function (or rather bringing together Newton's concept of fluxion and Leibniz' subtly different concept of the differential) and developing an algorithmic method for this calculation (the rules of derivation). However, to be able to achieve this, Barrow required analytic geometry in order to be able to transform, through handling curves (geometric objects) and using formulae (algebraic ones), the problem of tangents of curves into the problem of the derivatives of functions, as we would say today. Algebraic treatment was also essential in developing the rules of derivation. On the other hand, without having reduced the process for the calculation of tangents (derivation) to an easily applicable algorithmic method which could be inverted (what is now known as the calculation of antiderivatives), it was of little use to know that tangent and quadrature problems were inverses. Barrow failed to register the essential importance of his result. Barrow did not like the algebrisation of geometry practised by Fermat and Descartes, both French, something which would cost him the chance to become the father of calculus.

Calculus was finally born only when the two concepts are combined. To better understand them, let us refer to them as the precedents to our derivatives and integrals, algorithms developed for their calculation (the rules of derivation) and the inverse of the concepts established (what we now call the fundamental theorem of calculus). In order to solve the problems that had occupied Leibniz and Newton's predecessors, it now suffices to put these basic concepts into practice, suitably interpreted, using the algorithmic treatment permitted by the rules of calculus (as we saw in Chapter 1).

Chapter 3

Newton, the Last of the Magicians

The date 13 July 1936, represents a symbolic boundary between the before and after in studies on the life and works of Isaac Newton. That day, an auction began at Sotheby's for 332 lots containing part of Newton's manuscripts, as well as his correspondence and other documents and objects that belonged to him. The incredible history of Newton's manuscripts is strangely fascinating since it reveals to us the inner life of the Englishman.

The sheer volume of manuscripts, letters and other documents conserved by Newton is staggering, in spite of the fact that according to Newton himself, during the final stages of his life he managed to burn a large part of his correspondence and certain technical articles of an interior quality that he did not wish his successors to see. However, while this may have been the case, it does not rule out the fact that Newton was creating a legend of mystery and fantasy that would elevate him to mythical status. Just like the story of the apple, the incredible simplicity of which has been so useful in creating the popular perception of Newton as a genius, which he told William Stukeley shortly before death: "After dinner, the weather being warm, we went into the garden and drank tea, under the shade of some apple trees, only he [Newton] and myself. Amidst other discourse, he told me, he was just in the same situation as when formerly, the notion of gravitation came into his mind. It was occasion'd by the fall of an apple, as he sat in a contemplative mood. Why should that apple always descend perpendicularly to the ground, thought he to himself. Why should it not go sideways or upwards, but constantly to the Earth's centre? Assuredly, the reason is that the Earth draws it. There must be a drawing power in matter, and the sum of the drawing power in the matter of the Earth must be in the Earth's centre, not in any side of the Earth. Therefore does this apple fall perpendicularly or towards the centre [...]. That there is a power, like that which we here call gravity, which extends itself thro' the universe."

However let us continue with the story of the manuscripts. Newton died without making a will, and there were disagreements between the potential heirs, eight

in total, all of whom were descendants of the two daughters and a son Newton's mother had with her second husband, the protestant minister Barnabas Smith. With the exception of his preferred niece, Catherine Barton, and her husband, John Conduitt, the other heirs wished to make quick money from the inheritance, and as such, in July 1727, shortly after Newton's death, his library was sold to a certain John Huggins for £300, £30 above the starting bid.

Those documents and manuscripts which could not be sold were handed down to the Conduitts' daughter, also named Catherine, who married the Viscount of Lymington in 1740 and later to the 3rd Earl of Portsmouth who would give this record of Newton's legacy the name by which it is

The illustration shows the story of the fallen apple that allegedly inspired Newton's formulation of the law of universal gravitation

now known: the Portsmouth Collection. In 1872 the process of cataloguing them for the first time began and they were held at the University of Cambridge. The result was published in 1888, the year in which the documents were returned to the Portsmouth family, except for all the material related to mathematics, correspondence, books and other documents, which they bequeathed to the university.

The remainder, as we have already mentioned, was auctioned at Sotheby's in 1936, all of Newton's manuscripts on alchemy, chemistry and regarding the Royal Mint, all the material collected by John Conduitt for his planned biography of Newton, a large number of letters written by and addressed to Newton, notebooks from his youth, manuscripts on chronology, theology and the development of the infinitesimal calculus, two magnificent portraits and his death mask. Everything was sold in two days for just over £9,000. It is not hard to imagine the disillusionment of the ninth Count of Portsmouth who had put them up for sale because he was short of cash. The economist John Maynard Keynes purchased personal documents and manuscripts on alchemy, chronology, history and theology, and passed them to King's College at Cambridge; on the other hand, a large part of Newton's manuscripts on theology were acquired by the Orientalist Abraham S. E. Yahuda (who in fact exchanged some with Keynes) who handed them to the Jewish National and University Library in Jerusalem, where they arrived, after certain legal problems regarding inheritances, in 1966.

The 'hero of reason'

The intense study of Newton that was carried out following World War II, unparalleled in the study of any other scientist, can in fact be seen as an allegory of that auction, which had the virtue of making the practically untouched treasure of Sir Isaac's manuscripts available for the first time.

One initial consequence of this was a shift in the historical perception of Newton as a scientist and a person. The famous remark of John Maynard Keynes summed it up: "Newton was not the first of the age of reason. He was the last of the magicians, the last of the Babylonians and Sumerians, the last great mind which looked out on the visible and intellectual world with the same eyes as those who began to build our intellectual inheritance rather less than 10,000 years ago." This was the product of his study of the collection of Newton's manuscripts.

Faced with this great scientist, the father of modern physics, the discoverer of the law of gravity, the author of perceptive studies on the nature of light and col-

ours, the inventor of infinitesimal calculus, faced with the image of the hero of reason with a brilliant clairvoyance (a feature promoted by Newton himself), his manuscripts presented the contrast of a person who, for that very reason, was more complicated, more real. He was interested not only, not even largely, in scientific matters (although he did go through phases), but also in dark theological problems, the practitioner of an alchemy midway between the experimental and the mystical, and the author, in addition to *Principia* and *Opticks* of abstruse biblical chronologies, all of dubious scientific reason (even for his time), but to which Newton dedicated many more pages than to his science.

In the light of his impeccable and successful professional career, either as the young Lucasian chair at the University of Cambridge, as a smart member of Parliament, as a scrupulous functionary of the Treasury (no less than a Master of the Mint) or as the all powerful president of the Royal Society, the manuscripts auctioned at Southeby's reveal his unconfessable secret: a devout and resolute Arianism that accompanied him throughout his life since he was a child and which legally could have kept him from all his positions had it become known. Hence, in contrast to the detached sense of his reports on the most suitable alloys for minting British coins, the manuscripts reveal his diatribes against the Holy Trinity and are filled with fiery, surreal and pompous sections. The following quote, taken from one of Newton's many manuscripts on theology (in this case, comments on the prophecies), provide a good illustration: "And why the Church of Rome began to rule over the ten kings and tempt them with this idolatrous religion, thus becoming rich and powerful, from that point onwards is comparable with a woman cooked in purple and scarlet and covered with gems, who lives deliciously perched atop a throne on the seven hills and the horn of the beast in a desert of sterile spirituality, and fornicates with the kings of the land and makes nations drunk with the wine of her fornication and floods them with gold and silver and precious stones and pearls and fine purple lingerie and silk and scarlet and all the other expensive goods that enrich the merchants of the land with their stupefaciousness." It is still a paradox of his life that Newton, a strident antitrinitarian, was a member of the Holy and Undivided Trinity College during his time at Cambridge.

The Scottish sculptor Eduardo Paolozzi [illegible] for this sculpture of Newton [illegible] 1995

The complicated life of a genius

The first signs that pointed to Newton's complicated personality appeared three months before his birth, on Christmas Day 1642 according to the Julian calender (still used in England at the time) with the death of his father. With the passing of time (and according to Frank Manuel, author of a psychological study of Newton published in 1968, which is both interesting and occasionally disturbing), the father figure that disappeared before his birth would become none other than the figure of the Father God. Thus the entire course of Newton's life was a search for the truth, be this by means of science, theology or alchemy, his interlocutor being not his contemporary human beings, but the unknown father figure transmuted into God. Such mysticism makes it easier to understand the tremendous aggression displayed throughout his life towards criticism, however minor it may have been, of his scientific work. This resulted in the appearance of a visceral and absurd phobia every time it was proposed that he should publish one of his discoveries, causing enormous delays to the publication of his mathematical output. Certain treatises, in particular those related to infinitesimal calculus, were not published until up to four or five decades after they were written, whereas others were not published at all. All set the

A drawing of Newton's telescope included in a book of his letters that can be found at the home of the Royal Society in London. In the background there is a statue of the genius himself

scene for his dispute over the discovery of calculus with Leibniz, who was less fearful than Newton when it came to publishing the fruits of his intellect.

This was precisely the case with the publication of his first scientific work in 1672. In January of that year, Newton had entered the Royal Society of London after presenting his reflector telescope. The following month he published his first scientific work in *Philosophical Transactions*, a new theory of light and colours. The publication created enormous anticipation, not only in England, but also in the rest of Europe. As soon as it was published, the inevitable criticisms and disagreements arose regarding what Newton had claimed to prove in his work. Furthermore, it was not just anybody that disagreed or opposed Newton's theories of light and colours; critics included none other than Robert Hooke, considered the expert on optics, and Christiaan Huygens, the leader of European science took issue. Richard Westfall, author of the best biography on Newton, published in 1980, thus ex-

plained the consequences of the publication and the ensuing crisis: "The discussion that followed on the paper of 1672 tells us less about optics than about Newton. For eight years he had locked himself in a remorseless struggle with truth. Genius of the order of Newton's exacts a toll. Eight years of uneaten meals and sleepless nights, eight years of continued ecstasy as he faced Truth directly on grounds hitherto unknown to the human spirit, took its further toll. And exasperation that dullness and stupidity should distract him from the further battles on new fields in which he was already engaged added the final straw. By 1672, Newton had lived with his theory for six years—and it now seemed obvious to him. For everyone else, however, it still embodied a denial of common sense that made it difficult to accept. Their inability to recognise the force of his demonstrations quickly drove Newton to distraction. He was unprepared for anything except immediate acceptance of his theory. The continuing need to defend and explain what he took to be settled **plunged him into a personal crisis.**"

Manuel's psychological interpretation transforms Newton's challenge as a researcher (the search for the truth to which Westfall alludes) into a religious matter. God was his interlocutor identified with the absent father figure: "The corruption of a text of the writings and the failure of an experiment, or indiscretion in their interpretation constituted not only a violation of scientific method but sins, like making a false testament. Such lies were in many ways darker than crimes because they violated and sullied the truth of God's creation." Moreover, "scientific error was put on an equal footing with sin, because it was the consequence of laziness on his part and a failure in his service to the Divinity. For Newton, a sin was not an act of human frailty that could be forgotten, but a sign that the guilty party was pos**sessed by evil.**"

At the age of three, Newton suffered one of the greatest traumas of his life. He became separated from his mother Hannah Ayscough Newton when she married the cleric Barnabas Smith, who was sixty-three years old. The couple moved to Smith's rectory, a few kilometres from the Newton family home, where the young Isaac was left in the care of his maternal grandmother. The separation was traumatic and scarred Newton's personality, making him highly susceptible to any act that could be interpreted as trying to move him from where he felt he belonged, something perfectly consistent with the heated debates about priority he maintained throughout his life, particularly with Leibniz. Manuel describes the consequences of the separation from his mother, explaining that "Newton's mother is a central figure in his life [...]. Both were together during a crucial period and his

Engraving from the late 19th century showing the house where Isaac Newton was born,
Woolsthorpe Manor, close to Grantham, Lincolnshire

fixation on her was absolute. The trauma of her departure, the negation of her love, created anguish, aggression and fear. After total possession undisturbed by any rival, not even a father, as if he had been born of a virgin, she was taken away from him and he was abandoned. Some psychologists claim that the anxiety created by separation is more intense when it occurs between thirteen and eighteen months, while others cite an earlier period. Given that Newton was already thirty-seven months old when his mother remarried, the most serious period of damage had presumably passed. However the proximity of his mother's new home could have served to aggravate rather than alleviate the wound of the loss. The elegant bell tower of North Witham church stands tall and pierces the skyline, standing out from miles around. Hannah was there, scarcely a mile and a half away, with the Reverend Smith. The loss of his mother at the hands of another man was a traumatic event in Newton's life and one from which he would never recover. At any point thereafter, when confronted with the possibility that something of his would be stolen, he reacted with a violence that was comparable with the terror and anguish generated by this first and abrasive act of deprivation. He regarded all subsequent discoveries and positions he acquired as part of him and the mere threat of them being seized **plunged him into a state of anxiety."**

His mother returned to the Newton family home, once again a widow, in 1653. She brought with her the three children she had had with the Reverend Smith during their seven years of marriage, in addition to a few hundred books Newton inherited from his stepfather. They were largely related to theology and no doubt fed an interest Newton would cultivate throughout his life.

In one of the notebooks from Newton's manuscripts, there is a retrospective confession of sins committed prior to 1662 – when he was twenty years old –, the thirteenth and fourteenth read: "Threatning my father and mother Smith to burne them and the house over them." "Wishing death and hoping it to some." It is highly possible that when, in 1715, in reference to Leibniz and infinitesimal calculus, Newton wrote, "... second inventors have no right," the spectre of Reverend Smith, the "second" husband of his mother, lingered in his unconscious.

Newton spent a few years at the school in Grantham, located five miles from his home, where he moved when he was twelve. He lived in the house of the chemist, with whose step-daughter he fell in love. If what in all likelihood was the courtship of an adolescent, poorly suited to living with other members of his sex, can be referred to as romance, this would be Newton's first and last with a woman throughout his life.

The young Newton finally arrived at Cambridge at the start of summer 1661, having overcome, with the help of his mother's sister who had studied there, the reservations of his mother. Newton would remain at Cambridge for thirty-five years, during which time he would produce all of his science, despite possibly dedicating the majority of his time to other studies and activities: theology, biblical history and, above all, alchemy. Even though he was a genius, he also had a tremendous capacity for work, which he exercised throughout his life, and in particular during his time at Cambridge. There he did nothing other than work, work, work, forgetting on some occasions to eat or sleep, taking refuge in the solitude of his quarters, sometimes concentrating on his studies of optics, physics and mathematics, in which he would go on to make his impressive contribution to the scientific *corpus*. At other times, in fact the large majority, judging by the enormous quantity of manuscripts dedicated to such matters, he consistently made bold efforts to unravel the results of his experiments in alchemy, seeking reasons and arguments which would sharpen his Arian faith, always investigating the truth or, perhaps in the most authentic terms when discussing Newton, maintaining a constant dialogue with Father God. This continuous labour, this restless work – captured so clearly by Westfall in the title of his biography, *Never at Rest* – appears clearly reflected in Sir Isaac's manuscripts: "His manu-

scripts show that he made mistakes and learnt from them, that he took wrong turnings and failed to immediately understand the implications of his own ideas That is, the manuscripts reveal a human process that is understandable in a form not revealed by the sparkles of his brilliance."

Wooing science: *Principia*

In order to reveal the difference between the alleged sparks of genius through which a discovery takes place, without anyone's help, in as little time as it takes an apple to fall from a tree and the arduous, laborious and drawn-out process of conceiving of the seed of an idea, distilling it, separating what is essential from what is superfluous, perhaps even erroneous, aligning it with other ideas, and finally giving birth, laboriously and not without due pain, often aided by the previous discoveries of others, to what is now a discovery in its own right (the real process that lay behind Newton's work) we shall focus on the process that led Newton to one of his greatest scientific discoveries, the theory of gravity, and the composition of his magnum opus *Principia*. However, let us remind ourselves once more of Newton's tendency, particularly in the last years of his life, to emphasise the facet of visionary genius above the more prosaic, but nonetheless more realistic, one of the tireless worker. Herein lies the story of the apple which we have already mentioned, or this other quotation regarding the process by which he made his discoveries: "I kept the problem constantly in front of me and waited for the first hints of dawn to transform slowly and gradually into full and clear light." However on other occasions he was more realistic, in a letter dated 10 December 1692, he wrote that he owed *Principia* to: "nothing but industry and a patient thought."

At the start of the 1680s, Hooke joined with Christopher Wren, the famous architect and professor of astronomy at Oxford, and the young astronomer Edmund Halley to respond to the question naturally arising from Hooke's suggestion of central forces, what type of orbit would a planet follow if it were acted upon by a central force of attraction inversely proportional to the square of the difference? Of the three, Halley was the one with the best idea for finding the solution: it occurred to him that the best thing would be to ask Newton.

The visit took place in August 1684. The content of the meeting is revealed to us in Newton's confidences to Abraham de Moivre. The French exile who came to England for religious reasons, subsequently explained it as follows: "Dr Halley asked him what he thought the curve would be that would be described

by the planets, supposing the force of attraction towards the Sun to be reciprocal to the square of their distance from it. Sir Isaac replied immediately that it would be an ellipse. The Doctor, struck with joy and amazement, asked him how he knew it. Why, saith he, I have calculated it. Whereupon Dr Halley asked him for his calculation without any farther delay. Sir Isaac looked among his papers but could not find it, but he promised him to renew it and then to send it him..."

However, let us go back to Newton's *annus mirabilis*, the period of around twenty months between 1665 and 1666 spent at his childhood home of Woolsthorpe as a result of the epidemic of plague which forced Cambridge to close, when he first began to work on the problem and the scene with the apple took place. At that time, Newton had set himself the problem of planetary motion within the theory of Cartesian vortices, which he had studied previously on his own account during his years of education at Cambridge, starting, as Huygens had done, from a law of straight-line inertia and the gravity force centrifugal pair for the modification of

WAS THERE A THEORY OF GRAVITY BEFORE NEWTON?

The state of play regarding planetary motion in England in the years prior to Newton's arrival was as follows. The main figure was Robert Hooke, who would go on to become one of Newton's great enemies. Based on the fundamental principle of Descartes' straight-line inertia, Hooke replaced the centrifugal gravity force with the single principle of attraction as the cause of the inertia straight-line trajectory. In 1670 he presented his ideas in a lecture to the Royal Society, of which he was secretary, from 1677 to 1703. These can be summarised as follows:

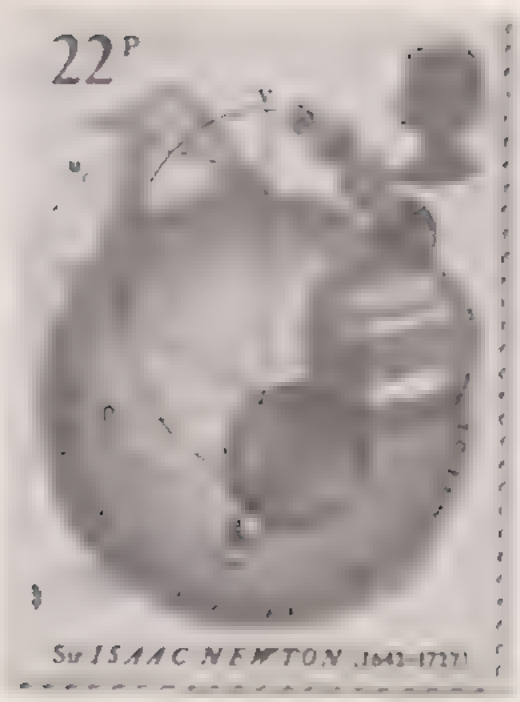
1. All celestial bodies have an attraction or gravitation towards their centre and attract the other celestial bodies which are within their radius of action.
2. Bodies move in a straight line except when affected by a force that causes them to follow other curved trajectories, such as circles, ellipses or any other more complex curve.
3. The power of forces of attraction diminishes as distance increases, according to a law which, at that time was unknown (years later by means of an analogy between gravity and light Hooke established that this attraction is inversely proportional to the square of the distance.

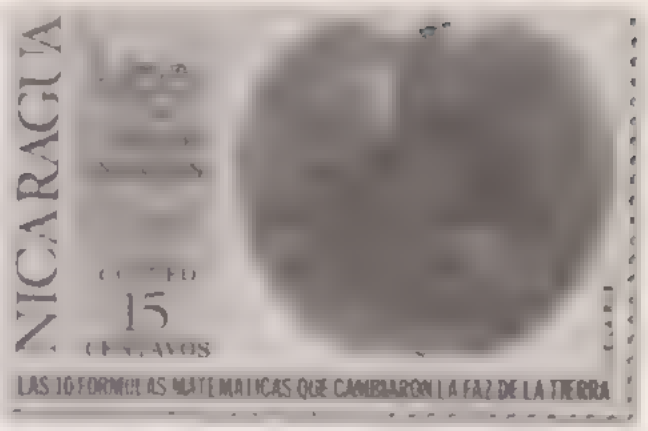
the straight-line trajectories. Combining these with Kepler's third law, he discovered that the centrifugal forces generated by the planets varied inversely with respect to the square of their distances from the Sun. Doubtless he already suspected at that point that the same gravity that made the apple fall was what kept the moon in its orbit around the Earth, but between this point and having the result fully developed, there stood a universe of effort, sleepless nights and intense work. In fact, he began by comparing the acceleration produced by the centrifugal force responsible for the Moon's motion with the acceleration of gravity at the Earth's surface. However, while the hypothesis was correct, he abandoned it since the calculations did not work out. (He had used imprecise values for the radius of the Earth and at that time was also unaware that the distances had to be measured from the centres.)

Newton did not continue his studies on the problem of planetary motion until ten years later. He was influenced by a letter received from Hooke in 1676, which asked his opinion on the hypothesis of considering planetary motion as a consequence of the law of straight line inertia and a force of attraction directed towards the centre of the orbit. This was the centripetal force, as Newton would later refer to it, that he would substitute for the centrifuge-gravity force pair. Hooke's query caused Newton to return to the problem of planetary motion and later on was also behind a monumental conflict between the pair, whom he accused of plagiarism while writing his *Principia*. As a result of his renewed attempts to tackle the problem, he discovered that Kepler's first two laws implied forces of attraction inversely proportional to the square of the distance. These were the calculations to which he referred during Halley's visit.

However returning to the course of events triggered by the visit. Newton, who had not lost his calculations, revised and completed them and, in November 1684, sent a small nine-page treatise to Halley with the title *De Motu Corporum in Gyrum*. Therein he provided the outline of a proof that the trajectory that would be generated by a force of attraction inversely proportional to the square of the distance would be a conic which, below certain speeds, is effectively just an ellipse (it also included the reciprocal result whose discovery, as we have just noted, was grounded in Hooke's letter).

Halley's insistence, together with Newton's genius and his overwhelming capacity for work would, two and a half years later, turn *De Motu* into *Philosophiae Naturalis Principia Mathematica*. When the manuscript was submitted to the Royal Society, they responded with the decision that "Mr Newton's *Philosophical Naturalis Principia*





Left, a British stamp in commemoration of the 300th anniversary of Principia *showing the elliptical motion of the planets. Above, a Nicaraguan stamp in which the solar system is contained in an apple, together with the formula for the law of universal gravity.*

Mathematica be printed forthwith in quarto in a fair letter." The edition, however, had to be financed by Halley from his own pocket, a setback, given the economic hardship suffered by the young assistant of the Royal Society at the time.

Principia is divided into three books, as well as a preface. Among other things, the work states Newton's three laws of physics. It is the third of these, *De Mundi Systemate*, in which he deduces the movements of the celestial bodies. He identifies the centripetal force that maintains the planets' elliptical orbits, with gravity. Consequently, the force that holds the moon in its orbit is the same as that which causes bodies with mass to fall to the Earth's surface. Furthermore, gravity is universal: every body in the Universe is attracted to the others in a way that is proportional to the product of their masses and inversely proportional to the square of their distances. Since this law implied those of Kepler's planetary motion, it was also possible to deduce that satellites complied with them in their motion around planets, as did comets with respect to the Sun, and the work also explained the disturbances caused by the universality of gravitational attraction which Newton studied for the case of the Moon. "At last we learn," wrote Halley in the ode to Newton appearing at the start of the first edition of *Principia*, "wherefore the silver moon once seemed to travel with unequal steps, as if she scorned to suit her pace to numbers – till now made clear to no astronomer."

A range of other issues were treated in *De Mundi Systemate*, including the theory that tides are an effect of the gravitational attraction of the Sun and the Moon

The title page of the first edition of Philosophiae Naturalis Principia Mathematica with corrections in Newton's own hand

on the seas, or the shape of the planets, necessarily flattened at the poles (a flattening that determined the period of their daily rotation), the proof of which inspired the French expeditions to Laponia, in Sweden and Peru in the first decades of the 18th century to measure the arc of the meridian, and resulted in the definitive triumph of Newton's system over the Cartesian one.

Newton and infinitesimal calculus

Isaac Newton was one of the most, if not the most, famous and celebrated scientists of all time. Although the fact is often overlooked, it is Newton, above all others, who most of all owes his hard-earned scientific fame to his mathematical ability and creativity. It was his talent as a mathematician and the discoveries which resulted in the differences that distinguished him from other scientists of his time, particularly in the preparation of his magnum opus *Principia*. In other words, Newton discovered the 'system of the world', which, according to Lagrange's pertinent observation, made him the most fortunate of scientists, since there was only one system of the world to be discovered. Furthermore, it was precisely the advantage Newton held over his contemporaries in the realm of mathematics that allowed him to consolidate

this discovery. In the words of Westfall, "Mathematics were, above all, Newton's first great intellectual passion. From mathematics he learnt criteria of rigour which would serve him for the remainder of his intellectual pilgrimage. Newton was to visit many strange seas of thought, speculative adventures from which more than one explorer of the 17th century never returned. The discipline that mathematics imposed on his fertile imagination marked the difference between wild flights of fancy and fruitful discovery." For those who think that Newton was a mere physicist (or perhaps it would be better to use the term natural philosopher), or even an applied mathematician, we should remember the figure behind the magnificent eight volume edition of Newton's mathematical manuscripts. D. T. Whiteside comments on the matter: "It should never be forgotten that for Newton mathematics had, above and beyond their place as a tool kit for the truth, an interior beauty and a vigour that was independent of all external motivations and applications. For those insensitive to the elegance and power of mathematics as an intellectual discipline in its own right, here you have a 'pure' mathematician, in the traditional sense of the word, on occasions completely absorbed in his ivory tower at Cambridge, developing theorems, properties, algorithms and constructions which were elegant in their own right. And how magnificently practical his talent and ability. In his time, there was no other mathematician better endowed or better versed, none that was more able in algebra, more masterly in geometry, more skilful or knowledgeable in the subtleties of infinitesimal variation."

Of all Newton's mathematical discoveries, the most significant, both in its own right and on account of the scientific returns it generated for him, was without a doubt infinitesimal calculus.

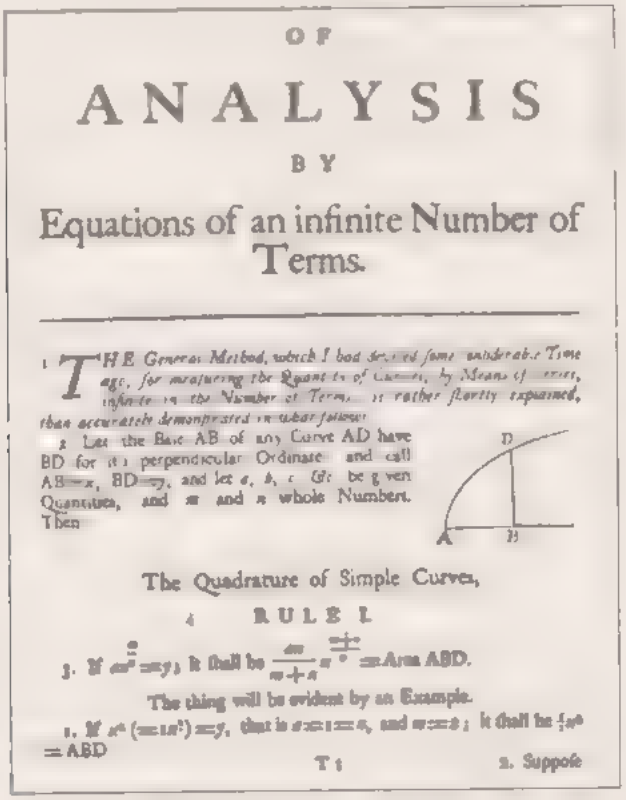
Newton's initial ideas on calculus date back to his *anni mirabiles* (1665–1666). In one of Newton's manuscripts dated a few years before his death in 1727, he wrote: "In the beginning of the year 1665, I found the Method of approximating series & the Rule for reducing any dignity of any Binomial into such a series. The same year in May I found the method of Tangents of Gregory & Slusius, & in November had the direct method of fluxions and the next year in January had the Theory of Colours and in May following I had entrance into ye inverse method of fluxions. And the same year I began to think of gravity extending to ye orb of the Moon and [...] I deduced that the forces which keep the Planets in their Orbs [...]." In fact, his first work on calculus was completed in 1669, despite not being published until 1711 *De Analysi per Aequationes Numero Terminorum Infinitas*.

Towards the end of June 1669, although the exact date it was written is unknown, in the space of a few days and based on research he had carried out from 1664, Newton wrote *De Analysi*. Its content was influenced by the logarithmic series published by Nicholas Mercator in his *Logarithmotechnia*, and the rumours and conjectures that would be made by Newton regarding the research others may have carried out or have been working on.

The content and approach of *De Analysi* is of fundamental importance and by making it public, at least in a restricted manner, Newton became the one to discover infinitesimal calculus, and *De Analysi* set out Magna Carta style what the new discipline meant. In the first part of the work, Newton showed how, using progressions of power series, it was possible to extend the calculation of quadratures to a wide range of functions, based on the basic quadrature of

$$ax^n.$$

It is perhaps worth explaining some of Newton's arguments here. In the interests of simplicity, we shall do so in the same way as Newton for the specific case of the curve which generates the area:



OF

ANALYSIS

BY

Equations of an infinite Number of Terms.

1. THE General Method, which I had devised some considerable Time ago, for measuring the Quantities of Curves, by Means of Series, infinite in the Number of Terms, is rather plainly explained, than accurately demonstrated in what follows.

2. Let the Base AB of any Curve AD have BD for its perpendicular Ordinate: and call AB = x, BD = y, and let a, b, c &c. be given Quantities, and m and n whole Numbers. Then

D

A B

The Quadrature of Simple Curves.

RULE I.

3. If $ax^{\frac{m}{n}} = y$, it shall be $\frac{an}{m+n}x^{\frac{m+n}{n}} =$ Area ABD.

The thing will be evident by an Example.

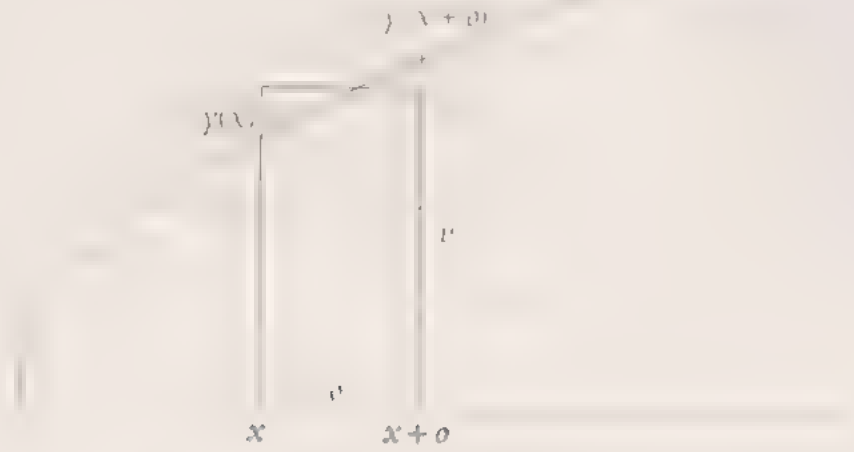
1. If x^0 (or $1x^0$) $= y$, that is $a = 1$, $= a$, and $m = 0$; it shall be $\frac{1}{1}x^1 =$ ABD

T 1 2. Suppose

First page of an edition of De Analysi.

$$A(x)=\frac{2}{3}x^{\frac{3}{2}}$$

Newton's method was as follows:



First, increase an infinitesimal quantity expressed as o on the x axis, such that the area is now increased to the section of the vertices x, $y(x)$, $y(x+o)$ and $x+o$, as shown in the figure. Now take the rectangle with sides o and v, such that the surface is equal to the increment in the area. This gives us the formula:

$$\frac{2}{3}(x+o)^{\frac{3}{2}}=A(x+o)=A(x)+ov=\frac{2}{3}x^{\frac{3}{2}}+ov.$$

Squaring and simplifying gives:

$$\frac{4}{9}\left(3x^2o+3xo^2+o^3\right)=\frac{4}{3}x^{\frac{3}{2}}ov+o^2v^2.$$

And dividing by o results in:

$$\frac{4}{9}\left(3x^2+3xo+o^2\right)=\frac{4}{3}x^{\frac{3}{2}}v+ov^2.$$

If we now make the increment in x infinitely small (i.e. we make o equal to zero) this gives $v=y$, and the previous formula becomes

$$\frac{4}{3}x^2=\frac{4}{3}x^{\frac{3}{2}}y, \text{ which gives } y=x^{\frac{1}{2}}.$$

Hence we can deduce that the area enclosed by the curve $y = x^2$ is $23 \cdot 32$.

We could be forgiven for thinking that Newton was only attempting to calculate the area enclosed by specific curves – in fact he went far beyond this. In this first part of *De Analysi*, he wished to show a general procedure and a certain value of abstraction that went beyond his initial interpretation of the calculation of an area: "Every problem on the length of curves, the quantity and surface of solids and the centre of gravity may ultimately be reduced to an inquiry into the quantity of a plane surface bounded by a curved line, there is no necessity to adjoin anything about them here", wrote Newton in a highly significant paragraphs, titling the section *Application of that Explained Above to Other Problems of this Type*. Newton used these paragraphs to distinguish between the first part of the treatise, where he set out the general method, and the second, where it was to be applied. We can agree that the result of his attempt is uncertain. Newton saw the enormous value of the abstract nature of the procedure, although perhaps, in this initial phase when the idea was still setting in his mind, he found it difficult to express and formulate. Perhaps also at that point he lacked suitable naming conventions and notation with which to refer to his creation. This declaration of principles was focused on the abstract problem of calculating a function given its derivative. Moreover, it established the inverse nature of the process with respect to the calculation of the variation of the function (as he would do at the end of the treatise) and finally provided an algorithmic procedure for the calculation of this variation (the derivative), although this would be minimal in *De Analysi*, with no clear rules, as Leibniz would later provide, for derivation. (Keep in mind that in *De Analysi* Newton did not exhaust all the material he had been building up on calculus prior to 1669.) All this makes it possible to conclude that infinitesimal calculus was already a reality with *De Analysi*. Indeed, *De Analysi* is a magnificent example which makes it possible to appreciate the supreme act of mathematical creation in all its glory: reading Newton's text allows us to bear witness to (in a certain way reliving) the birth of infinitesimal calculus in its Newtonian form. Hence when we immerse ourselves in the newborn calculus Newton sets out in the pages of *De Analysi* and attempt to recognise the properties of the adult calculus we have learnt beforehand, the sensation is like contemplating childhood photographs of somebody whom we have only known as an adult (our parents for example) and begin to recognise, beneath the youthful features in the images, the more familiar ones which have matured in the person that they have become.

Once *De Analysi*, which would make Newton's name known for the first time among British mathematicians, had been written, he showed it to Barrow who

naturally proposed it was sent immediately to John Collins, a member of the Royal Society of London who held the office of correspondence and was the channel for the communication and distribution of mathematical results and news, a sort of English Marin Mersenne. At this point, the unhealthy panic that gripped Newton when it came to publishing his works comes into play: exposing his ideas to the public to claim the paternity of his works also involved exposing himself to the critics. Here it is worthwhile pointing out, in order to clarify one aspect of the polemic between Leibniz and Newton, that in those days, the term 'publish' meant something quite different to today's usage of the word. Whereas nowadays it means to publish in a journal or in a book accessible to all those interested, at that time, when such channels (above all journals) were not as readily available as they began to be just a few decades later, the term 'to publish' also included the concept of distributing results in the form of a hand-written manuscript among a restricted group of interested friends, and in the case of Newton this group usually included someone dedicated to scientific dissemination, such as John Collins or, above all, Marin Mersenne.

However, in order to better appreciate Newton's apprehension, let us consider the series of letters exchanged between Barrow and Collins. At first, on 20 July 1669, Newton only allowed Barrow to inform Collins that he was in possession of *De Analysi*, prohibiting him from mentioning the name of the author: "A friend of mine here, that hath a very excellent genius to those things, brought me the other day some papers, wherein he hath sett downe methods of calculating the dimensions of magnitudes like that of Mr Mercator concerning the hyperbola, but very generall, also of resolving aequations; which I suppose will please you." Eleven days later, Newton agreed to let Barrow send Collins a copy of *De Analysi*, while retaining anonymity and requesting its subsequent return. Note the subtle way in which Barrow makes reference to reading and re-reading but not making a copy, an implicit way of saying that what he was sending was for Collins' eyes only: "I send you the papers of my friend I promised, which I presume will give you much satisfaction; I pray having perused them so much as you thinke good, remand them to me; according to his desire, when I asked him the liberty to impart them to you: and I pray give my notice of your receiving them with your soonest convenience; that I may be satisfied of their reception; because I am afraid of them; venturing them by the post, that I may not longer delay to corrispond with your desire."

After studying *De Analysi* and transmitting his enthusiasm to Barrow, permission was then given by Newton to reveal his name to Collins and allow others to see the

manuscript: "I am glad my friends paper giveth you so much satisfaction. His name is Mr Newton; *fellow* of our College & very young (being but the second veere Master of Arts) but of an extraordinary genius & proficiency in these things you may impart the papers if you please to my Lord Brouncker" (At that time, Lord Brouncker was president of the Royal Society.) With respect to the process, Westfall writes that "it was a fair show of apprehension by a man [Newton] who knew himself to be the leading mathematician of Europe."

Collins quickly returned the manuscript of *De Analysi* to Newton via Barrow, but not without having made a copy for himself. This copy, together with Barrow's letters, was finally discovered by William Jones in the batch of Collins' documents he purchased in 1708. Among other things, they helped him decide to suggest that Newton publish *De Analysi*, subsequently unveiled in 1711, and were also important when the dispute over the priority intensified, as independent evidence that would come to show that Newton was the first to discover calculus.

As Westfall notes, *De Analysi* had an important effect on Newton's professional life and probably helped him in his appointment to the Lucasian chair, created at Cambridge as part of the legacy of Henry Lucas. The stipend bequeathed by Lucas created one of the most coveted of academic positions. At that time, the Lucasian chair was the only one at the eight existing universities whose remit included, as we

NEWTON'S VERSION

It is interesting to consider Newton's version of the story in his own words. It can be found in *Epistolae posterior*. Newton's second letter to Leibniz. The letter contains a number of autobiographical paragraphs, the most interesting of which reads as follows: "When the ingenious work *Logarithmotechnia* by Nicholas Mercator appeared, I began to pay less attent on to these matters (power series and fluxional calculus), suspecting that Mercator[,] either knew how to [develop the series by the extraction of roots or [develop the series by the division of fractions or at the very least that others, upon discovering the development of the series by division, would be able to discover the remainder before I reached a mature age to write. Just as this book appeared, a compendium of the method for these series was communicated by Mr Barrow to Mr Collins, in it had indicated the areas and lengths of curves, and the surfaces and volumes of the solids generated by lines and, inversely, how to find the generator lines if they were given, and the method therein revealed that had shown this for various series."

would now say, mathematics and natural philosophy: the chair was responsible for giving classes on geometry, astronomy, geography, optics, statics and other mathematical disciplines, as well as submitting the text of at least ten of their lectures to the university library every year, with specified penalties if they failed to meet these terms. Newton broke these conditions quite frequently, although it appears he was never penalised.

In summer 1669, Barrow, who had held the Lucasian chair since its inception, five years earlier, was considering resignation. It does not appear, as is often claimed, that he was dazzled by Newton's genius; indeed, the decision was a result of other factors. More than a mathematician, Barrow was a theologian and wished to dedicate himself to his vocation. Furthermore, he also harboured ambitions of finding a better position, with more political influence, shall we say; indeed, in the years following his resignation, he was named the royal chaplain, and two years later became Master of Trinity College, a position which, according to the statutes of the Lucasian chair, was incompatible with that position, despite Barrow's attempts to avoid this incompatibility by requesting a royal dispensation. At any rate, Barrow tendered his resignation, and on 29 October 1669, Newton was appointed as Lucasian professor on Barrow's nomination.

For the remainder of 1669, Collins and Barrow tried to persuade Newton to publish *De Analysi*. There was nothing to be done. Westfall writes, alluding to the dispute with Leibniz: "Thus quietly did Newton's apprehensions sow the seeds of vicious conflicts."

Newton's second work, and his most important on infinitesimal calculus, *De Methodis Serierum et Fluxionum*, was written two years later and yet was not published until 1739. In it, Newton presents his concepts of the fluent (a variable considered as a function of time) and the fluxion of the fluent (the derivative of the variable with respect to the time): "To distinguish between the quantities I shall consider as perceptible but with indefinite variation, and the others that are known and determined in any equation, I shall denote the latter using the first letters of the alphabet *a*, *b*, *c*, etc., whereas the former shall be referred to hereafter as fluents and shall be denoted using the last letters of the alphabet, *v*, *x*, *y*, *z*. And the rates at which these flow and are modified by their generating movement (which shall be referred to as fluxions or simply speeds) shall be denoted using the letters $\dot{v}, \dot{x}, \dot{y}, \dot{z}$."

It is important to note (since as we have previously observed, this is why he can be regarded as one of the discoverers of calculus) that Newton presented the con-

cept of fluent and fluxion in their own right, as elements of a theory, and with algorithmic rules to facilitate their application in the calculation of the fluxion of a fluent. He then went on to apply them to solve problems of tangents, quadratures and maxima and minima. In the case of the latter, for example, he set out the following procedure: "When the quantity is the largest or smallest, at that moment its flow will neither increase or decrease: if it increases, this will prove it was lower and that what follows would be greater that the current value, and vice versa if it is decreasing. Thus calculating its fluxion, as has been explained, and equating it to zero." This describes the standard method currently used, calculating the derivative and equating it to zero.

On quadratures, he wrote "Problem 9: determine the area of a given curve. The solution to this problem is based on establishing the relationship between the fluent quantity and its fluxion," or rather, inverting the process for the calculation of fluxions (today we talk of inverting the process of derivation, or calculating the antiderivative). Hence, the fundamental theorem of calculus finally appears as an efficient tool for solving problems of areas.

To illustrate the power of his infinitesimal calculus, in *De Methodis* Newton applies it to almost all the problems for the calculation of areas, tangents, etc. which had occupied his predecessors for almost a century, solving one after another. However *De Methodis* was not published until a number of years after Newton's death. Why did Newton take so long to allow the publication of his first books on infinitesimal calculus?

As we have already remarked, the peculiarities of Newton's personality gave rise to a certain reticence when it came to publishing his results, a tendency which exacerbated the various polemics he maintained after publishing his first works. With respect to calculus, Newton's apprehension was greater still on account of the logical weakness it had in its early days. The concept of fluxion and the rules for its calculation, just like Leibniz' differential or the many infinitesimal devices of their predecessors, were based on so-called 'infinitesimal quantities'. These 'infinitesimals' were indefinitely small numbers, which were practically zero. They could be simplified when required, yet, since they were non-zero, could appear as the denominators of a quotient. Newton unsuccessfully attempted to avoid their use, and in fact in another of his works on calculus, *De Quadratura Curvarum*, published in 1704 as an appendix to his *Opticks*, his "evanescent increments" brought him close to the concept of the limit, which would be used by Bernhard Bolzano, and above all Louis Cauchy, early in the 19th century, as the foundations of infinitesimal calculus.

Print of Augustin Louis Cauchy, one of the most
[illegible] mathematicians [illegible]

Hence, conscious of the problem of the weaknesses in the logical foundations of his calculus of fluxions, Newton was especially reluctant to publish anything regarding it, although he always had a few copies of these works circulating in manuscript form among his friends. Without doubt, this fear also influenced the writing of his magnum opus, *Principia*. Newton opted for a Greek-style geometric language which was more secretive and darker, but more rigorous from a logical point of view; in fact, he eliminated almost every trace of his infinitesimal calculus that he probably, although we cannot be sure, used to obtain part of the results contained in *Principia*.

In any case, he did leave a snippet of his calculus in *Principia*, and this represented the first printed appearance, albeit in a highly literal form, of one of Newton's results on infinitesimal calculus. This took place in 1687, three years after Leibniz had published his first article on differential calculus in *Acta Eruditorum*. In specific terms, we are referring to Lemma II of section II, book II, which sets out equivalents to our rules for the derivatives of products and powers in a somewhat enigmatic manner. In fact, Newton proceeded somewhat artfully in order to avoid having to make the increments disappear, with this 'trick' being reported in the second half of the 18th century by Archbishop Berkeley, the greatest paladin in the crusade against the 'infinitesimal'. However, the appearance of this lemma (basis or step) was not the only contribution made by *Principia* to the history of calculus: there were others, of different types, which we shall cover further on when we tackle the bitter dispute between Newton and Leibniz over the the discovery of calculus.

We have previously seen how Newton makes reference to the discovery of his binomial theorem, or rather the development of $(1+x)^n$ as a power series, during his *anni mirabiles*. However, it was not published for the first time until ten years

later in 1676, and even then in a highly restricted manner. It was the first of the two letters that Newton would send to Leibniz through the secretary of the Royal Society, H. Oldenburg. In the letter, referred to as *Epistolae prior*, Newton set out, as we have mentioned, his binomial theorem and other results related to series, responding to previous questions from Leibniz.

The binomial theorem was essential for infinitesimal calculus. In fact, based on the binomial, he discovered the majority of progressions for elementary functions, those for trigonometric inverses (whose derivatives could be calculated using the binomial), and based on these, the trigonometric ones, and he calculated those of logarithms and exponentials in a similar way.

Some of these developments were already known, such as the case of logarithms, which appeared for the first time in the aforementioned *Logarithmotechnia* (1668) by Nicholas Mercator.

THE DERIVATIVE ACCORDING TO NEWTON

The proof provided by Newton in the *Epistolae prior* of the rule for derivatives of products, in today's terms, is as follows: "Any rectangle, as AB, augmented by a continual flux, when, as yet, there wanted of the sides A and B half the increments $\frac{1}{2}a$ and $\frac{1}{2}b$ (increments), was

$$A-\frac{1}{2}a \text{ by } B-\frac{1}{2}b, \text{ or}$$

$$AB-\frac{1}{2}aB-\frac{1}{2}bA+\frac{1}{4}ab$$

but as soon as the sides A and B are augmented by the other half increment, the rectangle becomes

$$A+\frac{1}{2}a \text{ by } B+\frac{1}{2}b, \text{ or}$$

$$AB+\frac{1}{2}aB+\frac{1}{2}bA+\frac{1}{4}ab$$

From this rectangle subtract the former rectangle, and there will remain the excess $aB+bA$. Therefore with the whole increments a and b of the sides, the increment $aB+bA$ of the rectangle is generated by (Q.E.D.). If we use the notation dA to write the increment of A, and dB to write the increment of B, something which there was no question of Newton doing since here we are using Leibniz's notation, we obtain the standard rule for the derivative of a product: $d(AB) = AdB + BdA$

A haughty genius

Newton was always extremely reluctant to offer his gratitude to others for any sort of inspiration or motivation that may have helped his discoveries, in spite of the extent to which he demanded recognition from others. Paradoxically, it is not uncommon to hear the following phrase attributed to Newton: "If I have seen further it is by standing on the shoulders of giants," as gratitude and acknowledgement of what he had learnt from others. Newton sent the phrase to Hooke in an exchange of letters which took place in 1676 and which permitted him to soothe, at least formally, the dispute that had taken place between them on the nature of light and its colours. The phrase can be traced further back to John of Salisbury, who, in his *Metalogicon* (1159), citing Bernardo of Chartres wrote: "We are like dwarfs sitting on the shoulders of giants. We see more, and things that are more distant, than they did, not because our sight is superior or because we are taller than they, but because they raise us up, and by their great stature add to ours."

The phrase can initially be interpreted as an expression of gratitude from Newton to Hooke, onto whose shoulders Newton had figuratively climbed in order to see further. However, there is also another, more devious interpretation, provided by F. Manuel, who reads this as a reference to Robert Hook's stature and slightly hunched posture: "Newton replied on 5th February, 1676, with a hackneyed image frequently quoted in literary struggles between the classicists and the modernists on the idea of process, an image which dates back to John of Salisbury and has often been cited, in the Name of Newton, out of context, as an example of his generous appreciation of his predecessors: 'If I have seen further, it is by standing on the shoulders of giants.' In the context in question, the psychological atmosphere of their 1676 correspondence, the meaning of the tribute is much more complicated, even ambiguous, it is a double-edged sword [ ] The image of a dwarf – not explicitly mentioned - raised up onto the shoulders of a giant sounds abruptly off key, until one comes to realise that there was something malicious in Newton's application of this vulgar simile to his relationship with Hooke. At first sight, it would appear that Newton is referring to Hooke as a giant and suggesting that he is merely a dwarf in comparison, however the hyperbole was, we should not forget, addressed to a man of limited stature and with a hunched posture, and therein lies a trace of mockery, conscious or unconscious, such as when a fat person is described as thin, thus highlighting their obesity." Westfall does not share Manuel's interpretation, because Newton was not so subtle in his attacks: "When he attacked, he bowed his head and charged."

Another indication of the repugnance Newton often felt when faced with acknowledging what he had learned from others can be found in his relationship with Descartes. From Descartes, he learnt the analytical geometry so crucial to the discovery of infinitesimal calculus. In spite of all that he had learnt from him, Newton came to profess a profound intellectual hatred towards the French mathematician. Revisiting Descartes' *Geometrie* in 1680, he filled the margins of the books with comments such as "I reject this", "error" and "this is not geometry". He even came to write an article entitled *Errors in Descartes' Geometry*. In fact, he ended up referring to analytical geometry as "the language of mathematical amateurs". In 1684, Newton even left a blank space in a manuscript in place of Descartes' name, as if he wanted to force himself to forget all that he had learnt from him: "On these matters I pondered nineteen years ago, comparing the findings of and Hudde with each other."

London: maturity and old age in the Treasury

Newton moved from Cambridge to London in 1696 to take up an appointment at the Royal Mint, first as warden and then in the most senior position of master (1699). Voltaire's malicious interpretation of this nomination is famous: "In my youth, I thought," he wrote in his *Philosophical Letters*, "that Newton owed his fortune to his enormous merits. I had supposed that the Court and the city of London had named him as the great master of the kingdom by acclamation. This was not the case. Isaac Newton had an enchanting niece, Miss Conduit, and the Chancellor of the Treasury, Halifax, liked her very much. Infinitesimal calculus and gravity would have been of little use without his pretty niece." Voltaire either exaggerated or was poorly informed, since when Newton was appointed as warden his niece was only seventeen and it is possible that Lord Halifax had not even seen her.

However it is true that later a strong relationship would develop between the two, which continued until Halifax's death in 1714 when Newton's niece inherited a fortune: "These gifts and legacies," wrote Halifax, "I leave to her as a token of the sincere love, affection, and esteem I have long had for her person, and as a small recompense for the pleasure and happiness I have had in her conversation."

Newton was an excellent but strict master of the Mint, throwing himself into the institution with all his intellectual brilliance, his enormous capacity for work and, without doubt, his already long and intense experience as an alchemist, much to the chagrin of many a minter of false coins who would be hanged as a result of the relentless pursuit to which Newton subjected them.

A number of portraits of Newton have survived, as well as masks and busts. Those shown in the photographs were by Godfrey Kneller. The one on the left shows Newton at the age of forty-six, shortly after publishing Principia, and the one on the right shows him at the age of fifty-nine, then Warden of the Royal Mint

A few years after his appointment to the Treasury, Newton received more scientific and social recognition: in 1703 he was appointed President of the Royal Society (a post which he exercised with hints of absolutism) and in 1705, Queen Anne knighted him at Cambridge's Trinity College.

Newton and his friends

Our profile of Newton would not be complete if we did not include something about the way he related to others.

Indeed, Newton's complex psychology made him a difficult person to deal with, and although in his last decades, which were spent in London, he had something of a reputation as a good host, this could well be down to the fact that visitors were charmed by his niece.

Perhaps for this reason, in his youth (let us say the 1660s and 1670s), Newton got on better with men who were older than him (e.g. Henry More, born 1614; John Wallis, 1616; John Collins, 1624; Henry Oldenburg, 1626; Isaac Barrow, 1630;

Christopher Wren, 1632, remember that Newton was born in 1642 than with his contemporaries, of which only a couple of names are known. Furthermore, his extreme puritanism made it difficult for him to maintain friendships: he broke off his relationship with John Vigani, an Italian who taught chemistry at Cambridge, for telling an immoral joke about a nun.

It is striking that the scientists with whom he sustained the most impassioned conflicts were all of his generation: Robert Hooke, born 1635; Leibniz, 1645; and John Flamsteed, 1646. Above all the confrontations with Hooke and Leibniz for matters of primacy and accusations of plagiarism and the one with Flamsteed (a good example of the abusive and absolutist way in which Newton often behaved towards the end of his life, once he had received his honours and recognition) were

NICOLAS FATIO DE DUILLIER (1664–1753)

Fatio was born in Basel, Switzerland, in 1664. His first passion was astronomy and in fact, for a while he studied with Cassini in the Paris Observatory. Unsuccessful in securing a position in the Académie des Sciences in Paris, he set off for Holland where, in 1686, he made contact with Christiaan Huygens, who recommended he study mathematics. Towards 1687 and now in England, Fatio had already prepared his own version of the infinitesimal calculus, even if it was lacking the sophistication of those of Newton and Leibniz. During the years that followed, he maintained a close friendship with Newton. However, after implication of his involvement with the Calvinists of Languedoc who were taking refuge in England, of which he was secretary, Fatio was disgraced, not only in Newton's eyes, but in those of almost all of European science. This remained the case until his death almost half a century later. He was offered the opportunity to save himself from ridicule by leaving England, but chose instead to follow the fate of his religious associates.

filled with crudeness and cruelty. Consider F. Manuel's comments with respect to Newton's conflict with Hooke. "When two powerful figures challenged each other in the small and close-knit scientific communities of the 17th century, the clash resounded. There was no place at the top of the Royal Society for both Hooke and Newton. In an almost animal way, only one of them could emerge as the champion of knowledge. Newton was an intruder in the kingdom where Hooke had once reigned supreme, and when the new star began to shine, Hooke began to lose power and status. He fought for his position, talking down the achievements of the new arrival, repeatedly accusing him of plagiarism, clutching at his post until his death in 1703 (just skin and bones towards the end). It was then that Newton would assume the presidency of the society, becoming the uncontested leader of the grey matter."

On the other hand, during his old age (from 1680 onwards), Newton 'preferred' the company of young men. Abraham De Moivre, born 1667, Fatio de Duillier, 1664, David Gregory, 1659, and Edmund Halley, 1656. Almost all of these were proteges Newton was appointing to chairs of British universities at the start of the 18th century. F. Manuel provides an interesting Freudian reading of this preference for young, talented figures: "At first, Newton identified himself with his mother who was thus transformed into a sexual object, he then discovered that young people were like him and he loved them as he would have wanted his own mother to love him."

Of all these, Nicolas Fatio de Duillier stands out on account of the polemic with Leibniz and what he meant to Newton.

Fatio arrived in England in the spring of 1687, in the early days of the publication of *Principia*, which had already aroused great expectations in scientific circles. He immediately fell under the influence of the new natural philosophy emanating from *Principia*, while showing a strong personal attraction towards the author, an attraction that was mutual. They met in 1689 and Newton's first letters to Fatio date from that year. Practically all scholars of Newton agree on the special nature of the tone adopted by Newton in these letters: the affection and warmth that impregnate them are not to be found in any other part of his correspondence.

The friendship between the two reached its climax towards the end of 1692 and the start of 1693, when Newton, after Fatio had suffered from fevers which almost cost him his life (or at least this is what he wrote to Newton in September 1692, offered him money and a room in his quarters at Trinity College, Cambridge. Fatio refused and, upon recovery from his fevers in 1693, set off for Switzerland to ar-

range some matters regarding his inheritance. Given that Newton suffered a profound mental crisis around the summer of that year, it is only natural to ask to what extent this was motivated by his relationship with Fatio.

The truth is that there are many hypotheses about the causes and circumstances that lay behind Newton's illness. Was it the result of stress and tiredness accumulated during the composition of *Principia*, a sort of post natal (or perhaps *post-Principia*) depression that belatedly but relentlessly began to set in? Was it caused by possible mercury poisoning, gradually brought on by his experiments in alchemy? Hair which allegedly belonged to Newton has even been analysed, and high concentrations of mercury have been found, although if the hypothesis of mercury poisoning was correct, he would also have suffered many relapses. Or was it just a simple case of depres-

Newton's tomb in Westminster Abbey

sion? In the end, perhaps it was an explosive combination of a little of each of these ingredients, doubtlessly including the stresses suffered by his intense relationship with Fatio de Duillier around the time. This is Frank Manuel's preferred hypothesis, with the concomitant connotations of (probably platonic) homosexuality, although many scholars of Newton, such as Whiteside, not only dispute this, but reject it outright.

The facts are that from 1693, a sort of rupture occurred from which Newton's relationship with Fatio would never recover, not even coming close to the periods of intense friendship experienced between 1689 and 1693. However, Fatio would make sporadic appearances in Newton's life: he was the first to publicly accuse Leibniz of having learnt the infinitesimal calculus from Newton.

The burial of Newton

Newton died on 20 March 1727, in London. In the words of Voltaire: "He lived honoured by his countrymen and was buried like a king who had treated his subjects well." Less florid prose than Voltaire's, are the words of Fernando Savater in *El Jardín de las Dudas*: "I was especially taken in by the funeral rites accorded to Newton. All of London took part in one way or another. The corpse was first of all displayed in a sumptuous catafalque, flanked by enormous torches, before being taken to Westminster Abbey, where it was buried among kings and other national heroes. At the head of the funeral procession was the Lord Chancellor, followed by all the ministers of the Crown."

Chapter 4

Leibniz, Master of all Trades

If Newton's production of unpublished manuscripts was seen as prolific, Leibniz was not far behind, consistently exceeding the former in certain aspects such as letters. Leibniz's manuscripts received better treatment than Newton's in the sense that they were not auctioned and that the process of editing, which was more or less systematic and studying them began almost a century before those of the English mathematician. However such study and editing have never reached the degree of intensity – and perhaps brilliance – achieved by the 'Newtonian industry' after World War II.

The final impulse for the complete publication of Leibniz's manuscripts and correspondence occurred at the start of the 20th century at the first conference of the International Association of Academies held in Paris. From that point onwards and with various interruptions as a result of the world wars, the ambitious task of editing them – still far from complete – has been undertaken. Of the eight series of Leibniz's complete writings and letters, the first three are dedicated to his correspondence. Leibniz corresponded with more than 600 people, a figure that includes a large number of his contemporaries in the fields of politics, science and philosophy. Over the course of his short life he was able to accumulate some 20,000 letters, an estimate of 10,000 of which were his own, with the others being addressed to him – compare this with the 1,000 that make up the seven volumes of Newton's correspondence. The majority of these are conserved, at least in draft form or in extracts, since like Newton, Leibniz was in the habit of keeping all his papers.

However, knowledge of Leibniz's manuscripts has not, as in the case of those of Newton, changed our image of the German genius a great deal, yet it has cleared up and helped to shed light on certain aspects of his intellectual production. In the opinion of Bertrand Russell, they have served to show us the best of Leibniz's philosophy, which, not conducive to improving his popularity among princesses or earning him money, was relegated to his drawers. They have also shed light (to give

an example relevant to this book – on how he came to discover infinitesimal calculus and how the process was independent of that followed by Isaac Newton.

The diversity and fertility of Leibniz's output reveals to us something already known by his contemporaries. He was a 'master of all trades' or, as the *Encyclopaedia Britannica* states, "one of the most powerful spirits of Western civilisation."

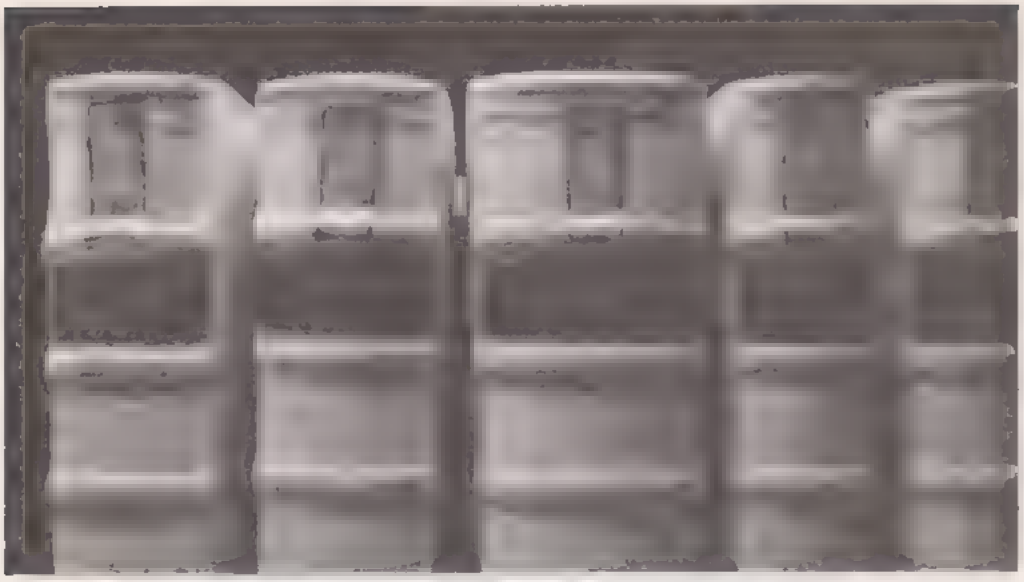
In a world like the one in which we live in today, which tends, perhaps even unwisely so, towards extreme specialisation, the contrasts of a mind such as Leibniz's are surprising. Here was a man who wanted to know about everything and who had something to contribute to all areas of knowledge, be it in philosophy, metaphysics, physics or mathematics, as well as in other activities that seemed further removed from those of the intellect (e.g. hydraulic presses, the drainage of mines using windmills, geology and linen production). However, in this ability to understand and have an opinion on everything, although not always with perspicacity and profundity, we can find certain central ideas that remain constant, such as his search for the *Characteristica universalis*, or universal language, which had to be symbolic and as precise as a scalpel, as well as the *ars combinatoria*, or deductive system that, in the words of Leibniz, would make it possible "to make [our reasonings] as tangible as those of the Mathematicians, so that we can find our error at a glance, and when there are disputes among persons, we can simply say: Let us calculate, without further ado, to see who is right." It was precisely his version of infinitesimal calculus, replete with its marvellous notation, a song to the search for a *Characteristica universalis* that gave order to the sea of results on quadratures, tangents, maxima and minima, centres of gravity, etc. We can find this echo of the universal in Leibniz's writings from the later part of his life, in which it becomes clear that, at the end of the day, his contribution to infinitesimal calculus was a language that allowed him to provide a unified method of handling the multitude of problems that had previously been handled using different methods.

They also reveal his obsession with universal harmony, reflected in his obstinacy in the task of reunifying the Christian churches, even though it was only to present a common face in the battle against the infidel Turks. Leibniz first attempted to unify Catholics and Protestants (he was Lutheran) and on a number of occasions resigned from positions as attractive as the head librarian at the Vatican or the Académie des Sciences in Paris due to their requirement for his conversion to Catholicism, and later, more modestly but with equally impossible results, tried to unify Lutherans and Calvinists. With this aim, he undertook intense public work that lay somewhere between theology and diplomacy, and did not cease to speak to different

A portrait of Leibniz in which he appears wearing a flamboyant formal wig of the time

parties, propose strategies and design procedures" (contrast this with the solitary and unknown Arianism of Newton).

All this came to nought – although unexpectedly, these failures would not reduce his boundless optimism in the slightest. Indeed, there was something inherently optimistic about being born in a Germany devastated by the Thirty Years' War and writing in his *Theodicy: Essays on the Goodness of God* "Amongst an endless number of possible worlds there is the best of all, else God would not have determined to create any [...] and you are at the source of happiness." "Our world, although it contains evil," explains Bertrand Russell on the Leibnizian principle, "has a greater abundance of good than evil than any other possible world, hence it is the best of all possible worlds and the evil it contains does not provide any argument against the goodness of God. This argument clearly appealed to the queen of Prussia. Her serfs continued to bear the evil while she continued to enjoy the good, and it was comforting that the great philosopher assured her that this was fair and legal." Voltaire did not need to exaggerate the caricature much when he decided to cast Leibniz in the role of the tutor Pangloss in *Candide*. "Pangloss taught metaphysics, theology and cosmology and proved by admirable means that there was no effect without a cause, and that in this world, the best that could be imagined, the castle of the lord baron was the most beautiful of all, and the baroness the best of all those that existed."



A number of volumes of the complete works of Leibniz, conserved in the library of the Benedictine Abbey in Göttweig, Wachau, Austria

It is fitting to provide a brief account of Leibniz's infancy, one not as 'rich' in trauma as Newton's. Leibniz did know his father, professor of moral philosophy at the University of Leipzig and also a jurist; his third wife was Leibniz's mother. His father died in 1652, when Leibniz was six years old. He left behind a formidable library, to which Leibniz only had access from the age of eight and which allowed the precocious child to educate himself, although he also attended school. In 1661 he began to study law at the University of Leipzig. Five years later, he attempted to claim his doctorate, but was rejected for being excessively young (or perhaps on account of his enmity with the dean of Law, as Leibniz would claim on other occasions).

He then moved to the University of Altdorf in Nuremberg, where he was able to earn a doctorate with a study on certain difficult legal cases. In Altdorf, Leibniz was offered a position at the university but declined. He never showed much enthusiasm for the universities of his time (perhaps justifiably so), instead supporting the creation of alternative institutions for advancing science and knowledge, including periodicals for the dissemination of ideas (such as *Acta eruditorum* that he helped to create in his home town of Leipzig), scientific societies, and academies, such as the one in Berlin, over which he presided after its inception in 1700, or the one in St Petersburg, whose creation, shortly after Leibniz's death owed much to the advice he had provided to Tsar Peter the Great.

It was possibly in Nuremberg in 1667 when Leibniz met his first patron, Baron Johann Christian von Boineburg, who accepted him into the service of the Elector of Mainz. The position allowed him to visit Paris shortly after, in 1672, where he was commissioned to propose an alliance to the French court of Louis XVI for the attack of Egypt, on the condition that France would halt the siege of Holland. In fact, his military version of the union of churches against the infidel served as a smokescreen for another, more prosaic mission: von Boineburg was owed rents and a pension in France, and Leibniz was also to investigate these matters. Regardless of his real purpose, he remained in Paris until October 1676 and the visit was fundamental to his education, above all in terms of science and mathematics, since it was during his last months in Paris that he discovered infinitesimal calculus.

Leibniz and infinitesimal calculus

"In the case of almost all the other important mathematicians," wrote J. E. Hofmann, the great scholar of Leibniz's thought from the first half of the 20th century, "the *great passion* can already be recognised during puberty and leads to new and decisive ideas in the period that immediately followed. In the case of Leibniz, this significant biological period passed without any special mathematical experience." In this respect, as in many others, Leibniz's education and career differ remarkably from Newton's.

Arriving in Paris at the age of 26, Leibniz hardly knew, and even then badly so, the first book of Euclid's *Elements*, and he knew little more than the arithmetic he had learnt at school (he had been taught from the book of the Jesuit Clavius). As he would admit to Johann Bernoulli (one of his first students) many years later, van Schooten's edition of Descartes' *Geometria* he had perused at university had seemed too complicated to him. In Nuremberg, where he arrived after completing his doctorate at the University of Altdorf (1666), he also had the chance to peruse Cavalieri's *Geometria indivisibilibus*. In spite of having mathematics in his blood, to borrow Hofmann's term, this was the pitiful mathematical knowledge he possessed upon his arrival in Paris in March 1672.

As in the case of Newton, a large number of Leibniz's manuscripts and documents have been conserved, in particular almost everything he wrote in Paris during his period of education, which Leibniz took great care to preserve. Through these documents, it is possible to provide a reconstruction of the trajectory of his education and the path to the discovery of his method of infinitesimal calculus.

During the first long year of his stay in Paris, Leibniz was quite the amateur when it came to mathematics. Indeed, he himself was later to recognise the mathematical ignorance from which he suffered at the time. He made his first visit to London during that first year and initiated his contacts with the English mathematicians through Oldenburg and Collins, however his "innocent ignorance" when it came to mathematics, which caused him to overestimate his own ability, coupled with his overly charming character, would result in more than one problem and misunderstanding with the Brits constituting the foundations of subsequent accusations of plagiarism.

Towards autumn 1672 he had yet to make contact with Christiaan Huygens, Europe's most famous scientist and mathematician, who was at the time employed at the Académie des Sciences in Paris. By that point, Leibniz had made his first mathematical discovery the addition of numbers using differences. Further down the line, he would insist that he perceived the inverse relationship between differentiation and integration in that inverse relationship between sums and differences.

Leibniz's argument was as follows: suppose we wish to add the numbers $a + a_1 + a_2 + \ldots + a_n$, and know that each of these is the difference of two others, i.e. $a_k = b_{k+1} - b_k$), hence a simple successive cancellation of the b_k means $a_1 + a_2 + a_3 + \ldots + a_n = b_{n+1} - b_1$.

Leibniz's inherent optimism and mathematical amateurism at that time led him to think that he had found a method which allowed him to add any series of numbers. This impression was reinforced when he recounted his discovery to Huygens, who, in order to test it, suggested that he add the series of the inverses of triangular numbers.

$$\frac{1}{2}+\frac{1}{6}+\frac{1}{12}+\frac{1}{20}+$$

Coincidentally, this series is one of the few that can be added through the application of Leibniz's procedure, given that the inverse of a triangular number has the form $\frac{1}{n(n+1)}$ and is thus the difference between $\frac{1}{n}$ and $\frac{1}{n+1}$, such that

$$\frac{1}{2}+\frac{1}{6}+\frac{1}{12}+\frac{1}{20}+\ldots = 1$$

From this point, Leibniz went on to add other similar series made up of pyramidal numbers, using the results he obtained to prepare a short treatise published in the

Journal des Savants, although this remained unseen since the journal went unpublished during 1673. In this treatise, Leibniz cited Cavalieri, Galileo, Wallis, Gregory, Pascal, Saint-Vincent and Archimedes as well as Hobbes as a great mathematician, indicative of an improvement in his education.

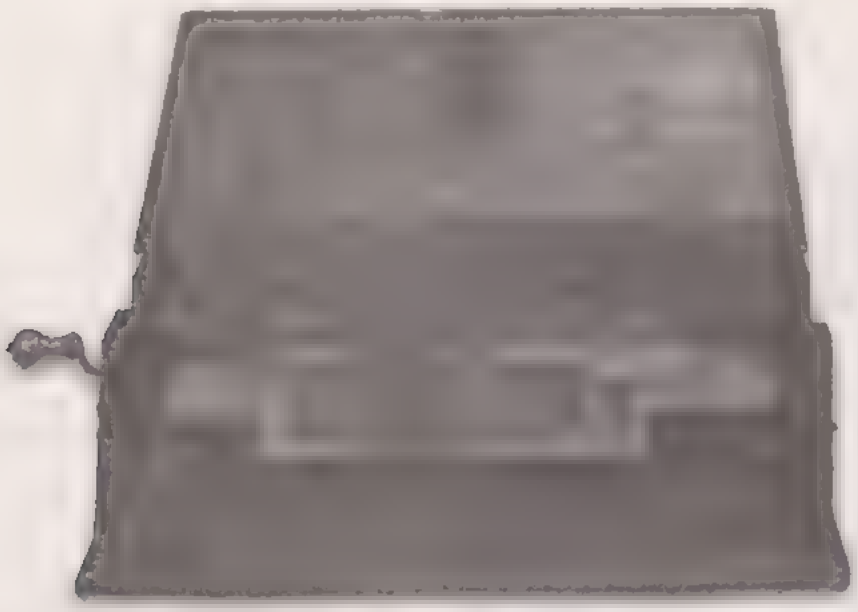
In January 1673, he visited London for the first time. He immediately met with Henry Oldenburg, compatriot and secretary of the Royal Society, who received him with open arms.

HENRY OLDENBURG (1618–1677)

Oldenburg was born in Bremen, Germany, although little is known about his youth. Towards 1654, he was commissioned by the Bremen government for two diplomatic missions to Oliver Cromwell's Britain. From then until 1661, the date he joined the Royal Society, he resided intermittently in England. Oldenburg is named as one of the secretaries of the Royal Society in the first two royal letters of constitution from 1662 and 1663, and remained in the post for fifteen years, until his death in 1677. Over the years, he created a complete system of archives and a network of scientific correspondents in Brit-

a letter addressed to a foreigner

In a session dated 22 January 1673, Leibniz presented the society with a wooden model of the calculating machine (which could perform addition, subtraction, multiplication and division) that he had prepared in Paris. In spite of the fact that it was far from perfect, he was admitted into the Royal Society. In April, Oldenburg wrote to him to confirm his acceptance as a member, although twelve months later, he was required to honour his commitment to present a more perfect model, although Leibniz would take years to do so.



Leibniz conceived a machine that could perform the task of multiplication through repeated addition

An incident between Leibniz and John Pell occurred shortly after, started by the Englishman. Leibniz met Pell while visiting Robert Boyle at his brother's house in Pall Mall. He later claimed that he visited Boyle on a number of occasions, since he "had no contempt for chemistry". Leibniz remarked to Pell that he had discovered a general method for representing and interpolating series using differences. Pell expressed his surprise since, given that Leibniz had come from Paris, he should have been aware that these results had already been published in France just a few years earlier by Gabriel Mouton; furthermore they were also known in England as well. The following day, Leibniz rushed to check Mouton's book in the Royal Society library to verify the truth of Pell's claim. Leibniz's version of finite differences is recorded in a letter he sent to Oldenburg on 3 February 1673 (communicated to Newton shortly after), meaning that, forty years later, when the dispute over the invention of infinitesimal calculus took place, Newton would write, as if wishing to

reveal a certain inherent tendency for plagiarism in Leibniz's work. "Pell accused Leibniz of having copied the interpolation method from Mouton's book."

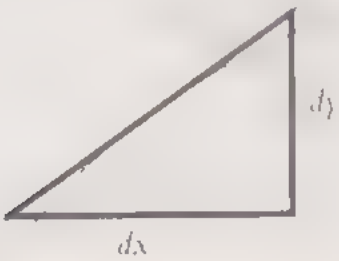
During the following months, an exchange of letters took place between Leibniz and Oldenburg in which he once more showed his amateurism and worrying lack of knowledge of the mathematics of the time. At that point, as Leibniz would later recognise, he knew nothing of geometry. Thus for example, in a letter dated April 1673, Leibniz communicated his results on the sums of the inverses of figurate numbers (mocked by Newton on account of being simple and straightforward), when Oldenburg informed him that his results could be found in the book *Quadratura Arithmetica* by Pietro Mengoli. Leibniz replied in error that Mengoli's solution only considered finite sums and not infinite series. Had he given more careful consideration to Mengoli's book, Leibniz would have been able to conclude and express in writing just what differentiated his results – the method used to obtain them.

The material communicated to Leibniz by Oldenburg included the results Collins deemed most representative of the discoveries made by the British mathematicians. They were collections of results, sometimes barely comprehensible and with errors made by the copyists, not methods for obtaining them and given that epistolary exchanges were often preserved in the records of the Royal Society, sought to guarantee the paternity of the results for Collins' compatriots. In fact, Newton made extensive use of these letters to ground his accusation of plagiarism against Leibniz, despite the fact that Collins' letters were not included in the information Oldenburg sent to Leibniz. In fact, Oldenberg translated them from English to Latin and often abridged their contents. This process of abridgement, together with the errors that were all too common in the transcription of mathematical results, rendered them almost incomprehensible.

In light of the material sent by Oldenburg, Leibniz realised that he would need to dedicate more time and effort to mathematics to complete his education. At that time, he had been trapped by his great passion for the subject. He broke off correspondence with Oldenburg for more than a year and set to the task. In the words of Hofmann: "He abandoned his relationship with Oldenburg to learn for himself and fill the gaps in his mathematical knowledge, of which he was now painfully aware. He did not resume the relationship until towards the summer of 1674. By that time, he was a changed man, a veritable expert in the field."

Leibniz would later write that he owed his introduction to advanced mathematics to the example and canon of Huygens. Upon the Dutch mathematician's

recommendations (who was currently in favour of Leibniz despite the pair having a rocky relationship), Leibniz latter studied Pascal, Fabri, Gregory, Saint-Vincent, Descartes and Sluse, to whom he added Mercator, whose book *Logarithmotechnia* he had purchased in London (together with Barrow's *Lectiones*, although it would appear that he studied these the following year). He found the books of the remaining authors in the royal library, although it is possible that on some occasions he purchased his own copy (this was certainly the case with van Schooten's edition of *Geometrie*, the book he had found so complicated back in Nuremberg). Of particular relevance was Pascal's *Traite des Sinus du Quart de Cercle*, from which he learnt to use what he would later refer to as the "characteristic triangle" a right angle triangle with its hypotenuse on the tangent to the curve and cathetii, the differentials of x and y respectively, as shown in the figure.



Years later, in a letter to Jakob Bernoulli, another of his students, Leibniz would observe that it was this work by Pascal which, like a bolt of lightning, made him clearly see that the problems of tangents and quadratures were inverses. He added that Pascal must have been blind not to have been able to see it for himself. Leibniz showed his calculating machine to a nephew of Pascal, possibly in June 1674 (Pascal also devised a calculating machine, although it only performed addition and subtraction.) After expressing his regret that some of Pascal's articles were still to be published, Leibniz succeeded in persuading the nephew to send him some of the manuscripts belonging to the French mathematician and philosopher.

Throughout 1673, Leibniz made use of the characteristic triangle to make a number of important discoveries, such as his method of transmutation (similar to the modern method of integration by parts), which, among other things, allowed

him to calculate the series for the arctangent from which he obtained his famous series for the number π. Based on this arithmetic quadrature of the circle, Leibniz came into conflict with Huygens in December 1673 regarding the classical Greek quadrature drawn using a ruler and compass.

After his studies and discoveries on quadratures, he studied the problems of tangents, for which his main source was the Sluse method. Based on a detailed study of Leibniz's manuscripts from that time, Hofmann has concluded that neither Newton nor Barrow was among his sources, just the names mentioned so far, to which Huygens should be added.

Leibniz then informed Oldenburg of the progress he had made in mathematics, partly owing to, in his own words, a "strange sort of luck" in a series of letters sent during the second half of 1674 and the first months of 1675. In particular, he informed him of the series for π, the series for the arcsine, and the method of transmutation, mentioned implicitly, although without providing details or formulae. Oldenburg's response showed him to be more critical of Leibniz than he had been at the start of their relationship, no doubt a consequence of the amateurish nature displayed by Leibniz at that time. After writing in a somewhat Sibylline manner, "I should draw your attention to the fact that the theory and method for measuring curves used by the aforementioned James Gregory and Isaac Newton can be extended to any curve, mechanical or geometric," he indicated to Leibniz, without providing details or formulae, a superior ability in British mathematicians, specifically Newton and James Gregory. In his letter dated 20 March 1675, Leibniz requested more information and details of these results, Oldenburg requested them from Collins and finally sent a letter to Leibniz on 12 April, which included Newton's series for sine and arcsine, Gregory's for the tangent and arctangent (as well as some of his interpolation series), and a number of other results on quadratures and other matters. (We should note that the letter included results but not the methods.) Upon receipt of the letter, Leibniz thanked Oldenburg for the series and, in Hofmann's opinion, without properly appreciating or understanding what he had been sent, promised to compare them with his own and then provide a better formulated explanation, something which he never did. Given that some of the series later reached him by another route, this was one of the reasons why Newton subsequently accused him of plagiarising his own, which had been sent via Oldenburg.

We can date the birth of Leibniz's infinitesimal calculus to the last days of October and the first of November in 1675, if it is not too bold, or perhaps incorrect, to attribute a date to a discovery such as calculus. The fact is that in the manuscripts

conserved from those days, especially those dated 29 October and 11 November, Leibniz introduced his notation for calculus. He used it to drive his algorithmic process that would subsequently define the differences from those used by his predecessors, investigate the rules which govern it and identify the processes of integration and differentiation as inverses. In the words of Hofmann: "Once the first and crucial step towards the 'algebrisation' of infinitesimal problems had been taken, a new vision revealed itself to a man who had experience identifying characteristic and general elements from among a jumble of similar elements. [...] He had a clear idea of how much was still missing from his calculus, but he knew that its defects could be overcome and that the path to a new world had been opened."

The key lay in the investigation of problems of inverse tangents, to which Leibniz had returned in October 1675, following his initial success the previous year in determining a curve given its subnormal and subsequent stagnation.

In a manuscript dated 29 October 1675, Leibniz introduced the $\int$ sign, a stylised representation of the first letter of the Latin word *summa*, to represent the operation of summing infinitesimals. Until then he had used Cavalieri's Latin abbreviation *omn.*, from *omnium*: all. In it he noted: "It will be useful to write $\int$ for *omn.*, such that $\int l = omn.l$, or the sum of the l."

Shortly after, in the same manuscript, he introduced the use of the letter d to denote differences. It was initially used on the denominator.

"This is obtained by the opposite calculation, or rather, assume that $\int l = ya$, let $l = \frac{ya}{d}$; now just as $\int$ grows d decreases the dimensions. However $\int$ represents a sum, and $\frac{d}{d}$ a difference." A few days later, in a manuscript dated 11 November, 1675, the d was moved above and $\frac{y}{l}$ expressed as dx. In this same manuscript, Leibniz raises the question whether $d(xy)$ is equal to $dxdy$, or whether

$$d\left(\frac{x}{y}\right) \text{ is equal to } \frac{dx}{dy},$$

concluding that this was not the case, although it would take him a long a time to find the correct formulae to differentiate products and quotients.

For Leibniz to be able to make out from among the jumble of his predecessors' geometric results on quadratures, centres of gravity, tangents, inverse tangents, etc., the common patterns that would lead him to synthesise the processes of integration and differentiation, the use of the algebraic language he mastered during his research into the solutions to equations carried out in the months prior to October 1675 was essential, algebraic research that was, after all, largely unfruitful and disap-

pointing, but that had the subsequent advantage of showing Leibniz how to use the powerful language of algebra, without which he would not have been able to invent his method of calculus.

Leibniz communicated the fundamentals of his method to Newton in his replies to the letters he received (via Oldenburg) in late and 1 October 1676. This exchange of letters between Newton and Leibniz is critical in the subsequent dispute regarding the priority of the discovery of infinitesimal calculus. As we have observed, Newton wrote two letters, one referred to as *Epistola prior* dated 13 June 1676, and the *Epistolae posterior*, dated 24 October 1676. Leibniz's replies are dated 27 August, 1676 and 11–12 June 1677. It is notable that in contrast to Newton's Leibniz's letters lacked their own names. In his letter, Newton wrote down for Leibniz a large part of the content of *De analysi* and *De methodis* on series, although little regarding his infinitesimal calculus. For his part, in his letters, Leibniz showed the advanced nature of his method of calculus. Newton must have realised that Leibniz had a method of calculus just as polished and powerful as his own. It would have been the time to publish his works to guarantee the paternity of the discovery. As Westfall writes: "Had he done so, we can only speculate on the consequences, however they would surely have been more honourable for both than the final outcome." Wallis was especially prescient: "I believe that Newton should perfect his notation and print these letters without further delay," he wrote on Newton's two *Epistolae*.

The house in which Leibniz lived during his time in Hanover

In terms of his working life, Leibniz's years in Paris were somewhat agonising. After the death of the Elector of Mainz in February 1673, and the cooling of hostilities between France and Holland, which caused him to lose the reason for his political/diplomatic mission in Paris, Leibniz began to

worry about being ordered to return to Germany, although his new patron offered to let him stay on in Paris without jeopardising his post.

Leibniz made a number of attempts to find work in the French capital, all of which were unfruitful, both in the diplomatic profession, where his low standing constituted an insurmountable obstacle, and in trying to obtain a paid position in the Académie des Sciences. At the start of 1675, he presented his calculating machine to the academy. However the number of foreign members (Huygens and Cassini among them) was too high for the French to contract another and thus, even though he used all of his contacts and influences in the second half of the year (which were by then quite numerous), and even attempted to obtain the Ramus chair of the Collège de France, left vacant after the death of Roverbal in October 1675, all his efforts were in vain. Time passed and the only job offer to arise was that of entering into the service of Duke Johann Friedrich, Elector of Hanover. Leibniz finally came to accept this offer, despite the fact that it meant he would have to live in Hanover, isolated from the main scientific centres and dependent on the favour of the elector, which there was always a high risk of losing. He succeeded in extending his stay in Paris as long as possible, first until May 1676 and then until October of the same year. However that was all. Having been guaranteed the post of librarian in Hanover (with his salary to begin in January), Leibniz left Paris for Germany on 4 October. He was never again to return to the city where, in his times of greatest pressure and in spite of his future in terms his career, he discovered infinitesimal calculus.

However, Leibniz did not return directly to Hanover, but travelled via London and Amsterdam. He spent ten days in London where, among other things, he visited Collins. "Dazzled by his visitor," Westfall explains, "Collins opened his archives." Leibniz read *De Analysi*, among other works, and even took notes, although only on series, which (according to Westfall) he saw was the material upon which the British mathematicians could instruct him. The absence of notes on fluxional calculus implied that he did not see anything he did not already know. After Leibniz had set off, Collins returned to reality and realised the scale of his indiscretion. He never told Newton what he had shown to Leibniz [...]. For his part, Leibniz chose not to mention it."

This was not the only thing Leibniz would abstain from mentioning when it came to what he had learnt on his return to Germany. In Amsterdam, he met with the philosopher Baruch Spinoza throughout the course of a month and discussed part of his *Ethics*, then still in manuscript form. Leibniz would later renounce his friend

Portrait of Baruch Spinoza. In spite of being a banned writer, his doctrine was highly influential on the philosophers that followed him, including Leibniz

At the time of his visit, Spinoza, who would die the following year, had already been causing problems among society for quite some time. He preferred neither to mention nor express his gratitude for all that he had learnt during their discussions, and nor did he wish to recognise the enormous influence of the *Ethics* on his subsequent philosophy on the issue.

When, twenty year later, the polemic for the paternity of the discovery of calculus began, what Leibniz did or did not see during his second visit to London was to prove crucial.

In 1677, now in Hanover, Leibniz had the correct formulae for the calculation of the differential of products, quotients and powers, formulae he discovered not without effort and after a process of trial and error.

In 1680, his method of calculus was practically complete and in marked contrast to Newton's reluctance to publish his work, Leibniz published his first article on calculus in 1684. This also constituted the first publication in the history of infinitesimal calculus. Its somewhat wordy title was *Nova methodus pro maximis et minimis, itemque tangentibus, quae nec fractas, nec irrationales quantitates moratur, et singulare pro illis calculi genus*. The article, just six pages long, sets out Leibniz's differential calculus in a highly schematic manner, without proofs and practically without examples. It was an extremely obscure and difficult work – "an enigma rather than an explanation", remarked the Bernoulli brothers, who were the first to learn Leibniz's calculus – furthermore, the difficulty was exacerbated by printing errors.

One of the few examples provided in the article, to which Leibniz dedicated the last lines, is worthy of mention. It is a solution to the inverse problem of tangents proposed by De Beaune: "As an appendix, I would like to add the solution to the problem that Descartes, having been proposed by De Beaune, attempted but was unable to solve. Find a line such that the projection of any of its points on an axis and the section of the tangent at this point with the axis form a segment with a

LOOPHOLES IN THE ARGUMENT

It seems fitting to reproduce the argument that led Leibniz to find the formula for deriving products since this provides a clear example of the logical weakness resulting from the handling of infinitesimal quantities. $d(xy)$ is the same thing as the difference between two successive xys. Let one of these be xy, and the other $(x + dx)(y + dy) = xy$; then we have $d(xy) = (x + dx)(y + dy) - xy = xy + xdy + ydx + dxdy - xy = xdy + ydx + dxdy$, and this is equal to $xdy + ydx$. We can omit the quantity $dxdy$, which is infinitely small in comparison with the rest, for it is supposed that dx and dy are infinitely small. As we can see, the logic of this deduction leaves much to be desired, revealing the most controversial property of infinitesimals, the fact that Leibniz considered the quantity $dxdy$ to be zero, in spite of the fact that neither dx nor dy are zero.

constant length." In half a dozen lines and in a clear demonstration of the power of his method, Leibniz found the solution using differentials – the logarithm.

We should also note the peculiar nature of the way in which Leibniz disseminated his differential calculus. He published his work in *Acta Eruditorum*, a scientific journal he himself had helped create in 1682, a sign of the new era being ushered in for science.

Leibniz would repeat this act two years later, in 1686, publishing a second article (also the second historical publication) on calculus, although in this case, integral calculus, in *Acta Eruditorum*. The title of the second article was *De geometria recondita et analysi indivisibilium atque infinitorum*. Leibniz began by justifying this new article on account of the difficulties in understanding the first. "It being understood that certain items published in these *Acta* for the advancement of geometry have not been sufficiently understood by certain learned men and moreover that they have been changed in their use and some have not been sufficiently understood, either as a result of an error in the writing or another cause, I thought it would be of great value to add in this space that which can be used to illustrate the previous matters."

It is in this article that his notation for integrals appears for the first time in printed form, although for typographical reasons, an *f* is used, the silent 's' of the time, instead of the ∫ symbol used by Leibniz in his manuscripts and, once the deficiency was rectified, for subsequent editions. The letter *d* also appears, to represent the differential, although Leibniz had already used this notation in his first article in 1684.

In the second, he insists on the inverse relationship between differentiation and integration, the fundamental theorem of calculus, ("like powers and roots in common calculations, sums and differences, or $\int$ and d are reciprocal", and he applied this to prove a theorem attributed to Barrow, for which he solved, as in the case of De Beaune's problem with which he closed the first article, another differential equation: "From what I have set out in the method of tangents, it is clear that

$$d\left(\frac{1}{2}x^2\right) = xdx;$$

hence the reciprocal is

$$\frac{1}{2}x^2 = \int x\,dx."$$

In terms of the name he chose for his calculus, Leibniz used the Latin word for difference, *differentia*, whence differential calculus, upon the suggestion of Johann Bernoulli, Leibniz would change his initial term *calculus summatorius* to *calculus integralis*. Johann Bernoulli also wanted Leibniz to change the $\int$ symbol to an "I", the first letter of the word 'integral'. In the end it was agreed to use Bernoulli's term integral calculus and Leibniz's $\int$ symbol. Hence in a letter dated 28 February 1695, Leibniz wrote to Johann Bernoulli "I leave it to your deliberation if it would not be better in the future, for the sake of uniformity and harmony, not only between ourselves but in the whole field of study, to adopt the terminology of summation instead of your integrals. Then for instance, $\int y dx$ would signify the sum of all y multiplied by the corresponding dx, or the sum of all such rectangles. I ask this primarily because in that way the geometrical summations, or quadratures, correspond best with the arithmetical sums or sums of sequences. I do confess that I found this whole method by considering the reciprocity of sums and differences, and that my considerations proceeded from sequences of numbers of lines or ordinates."

In addition to these two seminal articles, Leibniz published a number of others on infinitesimal calculus (Professor Cuesta Dutari estimates as many as 27 between 1684 and 1708 in *Acta Eruditorum* alone). As we have already noted, the discoverers of calculus also differed in this respect. Leibniz preferred to disseminate his findings in articles published in scientific journals (it was a faster and more flexible form of publication), whereas Newton preferred to publish them in book form (allowing him to delay the publishing process on a whim).

In the service of the Hanoverians

Upon his return to Germany, he no longer found himself in the service of the Elector of Mainz: the death of von Boineburg at the end of 1672, followed by the death of the Elector himself a few months later, left Leibniz practically unemployed. The best he could manage to find was a post at the library in the service of the Elector of Hanover which, on account of its duties, required him to live in the capital of Lower Saxony, far from Paris, at that time the cultural, scientific, and philosophical centre of Europe.

The Brunswicks of Hanover thus became Leibniz's patrons for the remainder of his life, although his career in the house of Hanover was not as prosperous as he would have liked it to be. In 1678, in addition to his post in the library, he also became a privy councillor, and in 1685, the family historian. However, the range of activities he undertook for his patrons was much greater. He advised on educational matters and he worked as an engineer and a geologist in the mountain mines of Harz, designing windmills for the drainage of the mines. This resulted in quarrels and economic hardship on more than one occasion, since the mills never came to work correctly, and Leibniz had to bear some of the costs. Part of the money to be generated by the drainage system was to be used for the creation of an academy for the development of the *Characteristica universalis*, which was sadly never founded. Leibniz also designed hydraulic dams, clocks and other mechanical equipment – not to mention his calculating machine.

Perhaps his greatest achievement for the Brunswicks was as a historian, tirelessly researching the old manuscripts of the Bavarian monasteries and Italian palaces, and even the epitaphs of lost tombs from unknown Carmelite cemeteries, he managed to show that his patrons were related to the House of Este, an ancient family from Modena. The research subsequently allowed the Brunswicks to stake their claim for and be awarded a new electorate in Germany.

Leibniz took his historical research to surprising depths, since not only did he uncover the story of the Hanoverians, but also he attempted to create a comprehensive history of the Earth, encompassing aspects as wide ranging as geological events and fossil records, and he investigated the migration of people through a study of the similarities and differences between different languages. Like so many of his other great enterprises, his vision of a genuine universal history never came to fruition, although the achievements and ideas he scattered throughout the ocean of his manuscripts and letters.

There is much to be gained by comparing Newton's history with that which Leibniz was fortunate enough to produce, especially when the comparison is extended to the circumstances in which each worked. How different Newton's solitary research into the ancient biblical kingdoms, searching for deeper knowledge of Father God, living a 'regulated' quasi monastic life at Cambridge, or later on in London, enjoying position and power, from Leibniz's hectic journey through archives and libraries (three years spent living at the south of Germany, Austria and Italy, plus innumerable other comings and goings between Berlin, Hanover, Wolfenbuttel and Vienna), tracing the ancestors of his patrons, patrons who would often become riled if they became aware that their family tree had not grown as much as they had wished.

Leibniz's relationship with the Hanoverians was good until, after the death of Ernesto Augusto in 1698, he had to deal with his successor (and son) George Louis. This future King George I of Great Britain never took to Leibniz, in spite of the magnificent relationships the philosopher maintained with the elector's

George I in a a portrait by Gottfried Kniller. His accession to the British throne meant a loss of recognition for Leibniz's work

HIS FASCINATION WITH THE EAST

Leibniz's interest in China, encouraged by [illegible] was [illegible] another example [illegible] and the [illegible] reached [illegible] that [illegible] the emperor [illegible] members of his family received from these missionaries

88 MEMOIRES DE L'ACADEMIE ROYALE

res Lineaires qu'on lui attribuë. Elles reviennent toutes à cette Arithmetique, mais il ſuffit de mettre icy *La Figure de huit Cova* comme on l'appelle, qui paſſe pour fondamentale & dont on ne ſçait rien, qu'elle eſt manifeſte, pourvû qu'on remarque premierement qu'une ligne entiere ——— ſignifie l'unité ou 1, & ſecondement qu'une ligne briſée — — ſignifie le zero ou 0.

Les Chinois ont perdu la ſignification des *Cova* ou Lineations de Fohy, peut-être depuis plus d'un millenaire d'années, & ils ont tort; car ces Caracteres leur ſeroient, où ils ont cherché je ne ſçay quels ſens éloignés. De ſorte qu'étant auſſi que l'extrême exactitude [illegible] nant des Europeens: Voicy comment. Il n'y a gueres plus de deux ans que j'envoyay au R. P. Bouvet Jeſuite Francois celebre, qui demeure à Pekin, ma maniere de compter par 0 & 1 [illegible] Ainſi m'ecrivant le 14 Novembre 1701, il m'a envoyé la grande Figure de ce Prince Philoſophe qui va à 64, & ne laiſſe plus lieu de douter de la verité de nôtre interpretation [illegible] Et comme ces Figures ſont peut-être le plus ancien monument de l'Ecriture qui ſoit au monde, cette reſtitution de leur ſens, après un grand intervalle de tems, paroîtra d'autant plus curieuſe.

Le conſentement des Figures de Fohy & de ma Table des Nombres, ſe fait voir lorſque dans la Table on ſupplée les zeros initiaux, qui paroiſſent ſuperflus, mais qui ſervent à mieux marquer la periode de la colonne,

Page from the article "Explanation of Binary Arithmetic" (1703–1705) which shows the figures of Fuxi. Leibniz believed that the binary numbering system had allowed him to decipher the meaning of these ancient symbols

mother and daughter, Duchess Charlotte and Princess Sophia Charlotte. The elector failed to appreciate Leibniz's intellectual qualities and only seemed to be concerned about his progress in the historical studies of his ancestry. Add to this the fact that he could never find Leibniz when he wanted to as a result of him taking advantage of any excuse to leave Hanover. He travelled to Berlin to attend to matters of the Academy of Sciences; he journeyed to Vienna to attend to the emperor (he was appointed aulic councillor of the Holy Roman Empire of the German Nation from 1712), he visited Tsar Peter the Great in Russia, taking advantage of the fact that he was passing through (he was privy councillor from 1711, with responsibilities for the promotion of mathematics and science in Russia and the reform of the law and the administration); and he was able to pay respects to other princes, princesses and electors, who united a Germany that was fragmented into a thousand kingdoms, bishoprics, principalities, cities, and states. He clearly got around.

However, despite the fact that George of Hanover did not appreciate the intellectual jewel that glistened in his household, he must in some ways have been jealous of the work to which this jewel was subjected by his neighbours and it must have tormented him, above all, that Leibniz would use any pretext to abandon the history of the Brunswicks and dedicate his intellectual efforts to any of the other many tasks on which he was working, be it the preparation of his latest philosophical works or his obsession with China.

The cordial and charming manner Leibniz often displayed throughout his life made it possible for him to win friends and cultivate relationships throughout the length and breadth of Europe. It is not that he lacked enemies or involvement in disputes, nor that on certain occasions his conduct lacked honesty, but his personality never displayed the complex psychology of Newton's

Leibniz's philosophy

Leibniz's philosophical system can be summarised in a handful of works (perhaps not as many as there could have been), the most significant including: *Meditations on Knowledge, Truth, and Ideas* (a short article published in *Acta Anditorum* in 1684), *Discourse of Metaphysics* (1686), *Theodicy: Essays on the Goodness of God, the Freedom of Man and the Origin of Evil* (1710) and *The Monadology* (1714), although a large part of his metaphysical and philosophical work (the best part, in the opinion of

Bertrand Russell) was not to be found in his books, but dispersed throughout his enormous output of letters.

His philosophy was based on a complex structure of principles. Some were related to reality: harmony, plenitude, continuity, the identity of indescernables. Others applied to the way in which reality was understood.

Of course the purpose of this book is not to explore Leibniz's immense philosophical output, but it would also be inexcusable to pass this by without dedicating at least some space to try to show at least one example of the uniformity that pervaded his work. Let us consider then, the principle of continuity. Such a principle is somewhat strange, coming from the father of monads, "nothing other than simple substances which make up compounds. By 'simple' I mean 'without parts'." Yet he had also conceived of the extension of a straight line as a compound of irreducible parts of infinitesimal length. Strange, even contradictory, it could be thought, but highly profitable since that was the principle of continuity which allowed Leibniz to pass from series of numbers to series of infinitesimals that formed a continuum, akin to passing from differentials to differential calculus. As Ferrater Mora explains: "The principle of continuity is a universal principle in which the harmony between the physical and the geometric is made clear. According to the principle, everything in the universe is related by virtue of metaphysical reasons, and not just in a present, but throughout time, since the present is always pregnant with the future. The principle of continuity makes it possible to give a reason for any reality and any event, since without such a principle it would be necessary to conclude that there are hiatuses in Nature, something which would be incompatible with the principle of sufficient reason."

The principle of sufficient reason also establishes that "it is impossible to find a true or existing fact [...] without there being a sufficient reason for it to be this way and not otherwise," this reason being understood as a cause and not a logical requirement. The principle of sufficient reason makes it possible to untangle the jumble of reasons to find the primary reason and source of pre-established harmony at the other end, or rather, God. Leibniz had a great love of proofs of the existence of God, proofs whose logic was dubious.

We also find the consistency and universality of Leibniz's intellectual fabric in an application of the principle of continuity to the flow of human migrations: if we admit that continuity is present in the development of human language, argues Leibniz, a discontinuity in language would indicate that migration has taken place

LEIBNIZ AND MARRIAGE

[illegible] Leibniz was not such a puritan as Newton, and [illegible] when he reached the age of fifty-one thought of getting married. However, the lucky lady took longer than she should have to decide, by [illegible] This story was told by Fontenelle, [illegible] and more than one [illegible] the time for marriage came. For [illegible] don't know [illegible] calculated at all, [illegible] could never decide. As [illegible] marriage, and that [illegible] to do so.

The burial of Leibniz

The honours paid to Leibniz following his death on 14 November, 1716 were markedly different from those shown to Newton. Perhaps this can be explained by the poor relationship between Leibniz and George of Hanover, which was aggravated after the disappearance of his sister Sophia Charlotte in 1705, then Queen of Prussia, and her mother, the Electress Sophia, in June 1714.

Leibniz returned to Hanover in the second half of the year when the king had already set off for England, after having spent more than a year and a half in Vienna. Leibniz would have liked to accompany his patron to England when he acceded to the throne. It was the toughest phase of his dispute with Newton on the discovery of calculus, and Leibniz made a veiled suggestion of the pleasure it would give him to present himself in the very country where he was held as an enemy, protected and integrated into the court of its king (and this was feared by more than one of Newton's supporters). He repeatedly requested the position of historian of England to no avail.

Even though, in 1700, Leibniz would have excellently defended the Hanoverian claim to the English throne, George I did not wish to see his historian in England where he could be distracted by fights with the natives about infinitesimal calculus while his family tree withered from lack of attention. As such, he ordered

him to stay in Hanover, exiled in his own land, practically under house arrest after being explicitly forbidden from making long journeys (1715) and had his salary withdrawn for two and a half years after Leibniz over extended his stay in Vienna. The order was categorical. Leibniz was to expend his efforts on the work the king desired so badly, the *Annales Imperii Occidentis Brunsvicensis*, a project that was never completed and which for Leibniz, as he himself would recognise, constituted a **genuine Rock of Sisyphus.**

As such, he was buried without pomp and with hardly any splendour, to the hymns of a children's choir and surrounded by his relatives and closest acquaintances, without the presence of a representative of the court, despite the fact the king and his entourage were hunting on a nearby farm at the time. For more than half a century, the tomb went without an inscription, in spite of the fact that his only nephew had inherited a considerable sum of money from Leibniz, to say nothing of the House of Hanover that he had served so well for so many decades.

Chapter 5

The Dispute over Priority

In this chapter, we shall turn our attention to the harsh and unpleasant dispute between Newton and Leibniz (and their respective followers) over who was to be recognised discoverer of infinitesimal calculus.

To do so, let us begin by considering the differences, albeit in a general manner, between Leibniz and Newton's methods of calculus in order to be able to gauge to what extent they were conceptually different.

From 1666, Newton was considering curves (fluents) as generated by a point in motion. He then came up with the concept of fluxion (derivative with respect to time). Note that the fluxion of a fluent at a given moment, or rather the instantaneous speed, would once again be a number. Newton then developed the corresponding algorithms for the calculation of fluxions (the equivalent to the rules of our calculus for the derivation of sums, products, quotients etc.), and showed that to calculate the area that encloses a curve, it suffices to calculate one of its fluents by fluxion (calculate an antiderivative, as we now say) and apply the fundamental theorem of calculus. It is precisely here that the usefulness of power series becomes apparent. To calculate the fluent of the fluxion, the latter is expanded as a power series and integrated term-by-term, using the rule for the integration of powers (crudely and anachronistically speaking).

Leibniz, on the other hand, considered curves to be polygonal lines made up of straight lines of infinitesimal lengths, whose extension created the tangent for each point on the curve, and from whose geometry, described through the algebraic relationships explicit in the formula that defines the curve, we obtain the corresponding relationship between differentials. He defined the concepts of differential and integral (rather than 'defining', perhaps it would be better to say that he 'discovered their operation', in contrast to Newton, for whom the differential of a function was an infinitesimal quantity). He showed that they are inverses (the fundamental theorem of calculus). Finally, he developed the corresponding process for the calculation of differentials (the rules of derivation), and the differentials of

elementary functions. He dealt with this final issue in a way that was much more symbolic than Newton: Leibniz preferred closed expressions over and above expanded series. A large part of Leibniz's contribution can be found in this analytical procedure.

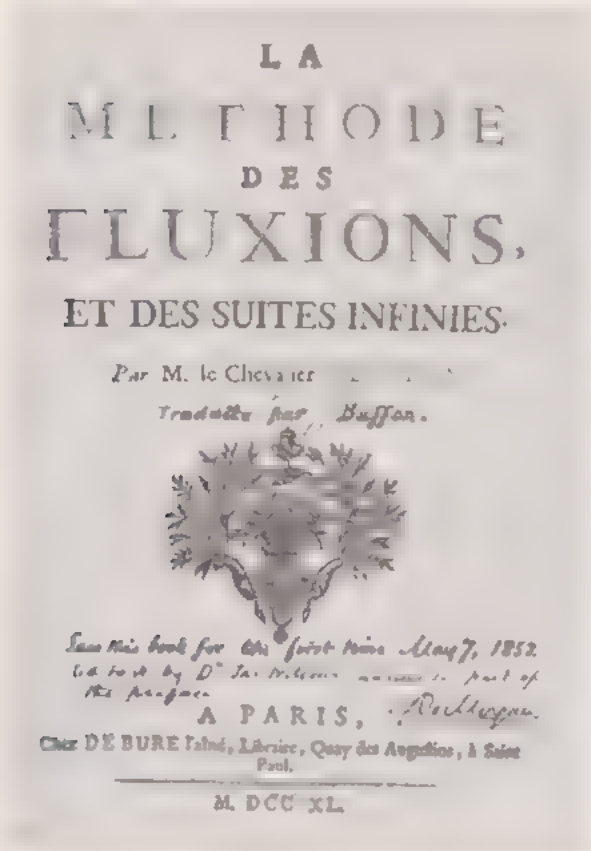
In terms of the difference between their methods of calculus it is significant that Leibniz always separated the discovery of the expansion of power series from the discovery of infinitesimal calculus: he attributed the first to Newton without any problems and even during the most difficult part of the dispute, yet he fought hard to claim the second as his own invention, even claiming that Newton had developed it based on the letters he wrote in reply to the *Epistolae prior* and *posterior*. In contrast, Newton always insisted that both the use of the expansion of series and infinitesimal calculus went together and that Leibniz, having learnt the former from him, was also indebted to him for the discovery of differential calculus.

Initial mutual acknowledgements... although lacking sincerity

Although Newton was the first to discover and develop his calculus, a decade before Leibniz, the latter was first to publish. In the first of his articles (1684), Leibniz did not mention Newton, although he would do so in the second (1686): "In order that it does not seem that I am attributing too much to myself or underestimating the work of others, it is only fitting that I take a few words to acknowledge what in my formulations owes much to the distinguished mathematicians of our century in this area of geometry [...] Moreover, Nicholas Mercator, from Holstein, mathematician and highly learned man, who was the first to my knowledge to give a quadrature as an infinite series. Furthermore, not only did he make the same discovery independently, but also perfecting it with a universal reason, a profoundly ingenious geometer, Isaac Newton, who, if he published his thoughts, those which I understand him to have, he would without doubt provide new paths for extraordinary increases and works in science."

Newton made reference to Leibniz at the first opportunity available to him. This was none other than the publication of the first edition of *Principia* in 1687. As is well known, Newton preferred to use the synthetic geometric language in the Greek style for the mathematics in *Principia*. At the decisive moments of the dispute, he frequently claimed to have used his fluxional calculus to deduce a large part of the results in *Principia*, despite then subsequently presenting them in a geometric language. However, while this may have been so, there is no documentary evidence: as

Whiteside has remarked on more than one occasion, the manuscripts in which Newton, according to his own account, used the fluxional calculus to deduce the content of *Principia* have not been found. However, as Westfall notes, "The problem with the mathematics of *Principia* lies not in finding previous proofs carried out in a different way, but in discovering the patterns of the calculus behind the geometric facade." Hence there is nothing explicit in *Principia* regarding Newton's sophisticated fluxional calculus, with the exception of Lemma II of Book II, which we have mentioned in Chapter 3. There Newton succinctly sets out what we would today refer to as the rules of derivation. To this lemma, Newton added a note in which he referenced Leibniz and made an explicit claim regarding the priority over him on the discovery of calculus. This was Newton's reaction to Leibniz's first publication on infinitesimal calculus. The note read as follows: "In letters exchanged between myself and the most excellent geometer G. W. Leibniz, ten years past, when I confirmed to him I had discovered a method for determining maximums and minimums, drawing tangents, and other similar tasks (and for both surds [square roots and other roots] and rational numbers), and when I masked it using transposed letters that contained the phrase *Given an equation which contains as many fluents as desired, find the fluxions and vice versa*, the most distinguished gentleman replied that he had also found a method of the same nature and communicated his method to me, a method which differed only slightly from my own except in its naming conventions and notation. The basis of both methods is contained in this lemma." Newton could have written the note to stake his claim to the paternity of calculus, however, up to that point he had not published anything on the matter (whereas Leibniz had). Except for an extremely small group of Newton's close acquaintances, the letters


LA
METHODE
DES
FLUXIONS,
ET DES SUITES INFINIES.
Par M. le Chevalier
Traduite par Buffon.
A PARIS,
Chez DE BURE l'aîné, Libraire, Quay des Augustins, à Saint Paul.
M. DCC. XL.

French edition of the Method of Fluxions *dated to 1740*

exchanged with Leibniz were unknown, so the note was understood as Newton giving Leibniz credit as an independent inventor of infinitesimal calculus.

Towards the end of 1691, four years after the appearance of *Principia*, the first reproaches against Leibniz regarding what he may have learnt from Newton began to circulate amongst those close to them. Fatio de Duillier wrote to Huygens: "Mr Newton is without doubt the first to have discovered the differential calculus of which you are now aware, prior to Mr Leibniz even having any idea about what he knows today, moreover, he was only able to have this idea after seeing what Mr Newton had written on the subject. Hence I am extremely surprised that Mr Leibniz has not indicated anything to this effect in the Leipzig *Acta*." Fatio returned to the matter on 5 February 1692: "I have no doubt that the publication of the letters [*Epistolae prior* and *posterior*] is offensive to Mr Leibniz, since it was only quite some time after having received them that he published the rules of his own differential calculus, and did so without mentioning his debt to Mr Newton. Furthermore, the manner in which he presented them was as a re-writing of what Newton had. Comparing them, it is impossible to avoid the clear sensation that the difference between the two is that of a perfect original and a shoddy and imperfect copy." Fatio's words make it possible to consider him as an instigator of the dispute.

THE INITIAL APOLOGIES

In 1693 there was another exchange of letters between Newton and Leibniz, in this instance [illegible] a letter from Leibniz to Newton dated March 1693 and most likely prompted by the news he had received, from Huygens regarding Newton and his publishing intentions and Newton's letter dated October 1693. The exchange was brief and from a scientific point of view cannot be compared to the previous one in 1676. A couple of paragraphs from Newton's letters stand out. First of all, consider his student greeting at the start: "I have not replied to your letter because it slipped through my hands, it was lost among my papers and I did not come across it until yesterday. This upset me because I greatly appreciate your friendship and for many years I have considered you as one of the foremost geometers of the century as I have had occasion to acknowledge when the occasion has arisen." The first paragraph concludes with a solemn declaration that he would be responsible for degrading [illegible] "I hope I have not written anything that displeases you, and if there is something you believe merits having been censured please advise me since I value friendship more highly than mathematical discoveries."

although it is extremely possible that the opinions he sent to Huygens were what he heard Newton say at that time.

Five years later we can find similar reproaches, however this time regarding Newton. In a letter dated 15 August 1696, Johann Bernoulli wrote to Leibniz: "I do not know if Newton invented his own method after having seen your calculus, especially when I see how you shared it with him before he had published his own method." This was the first malevolent suggestion that he made to Leibniz regarding the possibility of Newton having committed plagiarism.

"Modesty is a virtue, but diffidence is a fault"

From 1691, the recommendations that Newton should publish his work were growing increasingly frequent. Of these, the most persistent and intense was from John Wallis above all, in asking him to publish the entire text of the *Epistola prior* and *posterior*.

In 1695, he wrote to Newton, informing him of the public recognition Leibniz had received for his method of calculus: "I have been made aware from Holland by means of your friends there, that the *fluxions* of yours have enjoyed success, receiving great applause in the guise of *Leibniz's differential calculus*. [...] You do not take sufficient care of your reputation (nor that of the Country), as you should, by allowing items of value to rest so long by your side until others take the credit which corresponds to yourself. I have achieved justice on this matter and now regret not having published these two letters word for word." Wallis continued in his insistence: "I

The letters of Newton and Leibniz published by John Wallis gave Newton priority in the development of his method of calculus

own that modesty is a virtue, but too much diffidence (especially as the world now goes) is a fault." Two years later Wallis told Newton that he himself was going to publish the *Epistolae prior* and *posterior* unless he expressly forbade him from doing so and, in fact, the letters finally appeared published in their entirety in 1699 in one of the volumes of Wallis' mathematical works. This also included copies of Leibniz's letters in response to Newton's; permission to publish Leibniz's letters had previously been obtained with considerably less effort than that required for Newton.

The letters published by Wallis changed the situation completely. Despite lacking precision, they showed the development of the methods of infinitesimal calculus used by Newton and Leibniz around 1676. However, what was important was that for the first time, documents were published that testified to the fact that although Leibniz had published before Newton, Newton had been developing his calculus prior to him and had even disclosed it, even if only partially and obscurely, to Leibniz when asked. In the summer of 1699, Leibniz wrote "Wallis has requested my permission to publish my old letters [...] as I have nothing to fear [...] I have responded that he may publish all that he sees fit to do so." Soon after, it would become clear that Leibniz was very much mistaken when he said he had "nothing to fear".

"You can tell the lion by its claw"

During those years, another incident occurred to favour confrontation. This was the famous challenge set by Johann Bernoulli in June 1696 regarding the brachistochrone problem: the problem was to determine the curve according to which a body falls, in the least time possible, between two points that are not in a vertical or horizontal position, where it is only acted upon by gravity. In May 1697, Leibniz was responsible for publishing the solutions that had been received. Four in total, from Leibniz, the Marquis de l'Hôpital, Jakob Bernoulli and the person proposing the challenge, Johann Bernoulli. However, another solution by an anonymous author was also published, having first appeared in *Philosophical Transactions* in 1697. As is well known, the anonymous author was Newton: the fact that he explained his succinct solution in just seventy-seven words was enough to allow Johann Bernoulli to guess the author: *"tanquam ex unguc leonem"* ("you can tell the lion by its claw"). All the solutions, except the one from the Marquis de l'Hôpital, identified the curve in question as a cycloid.

The rest of the story, provided below, was revealed in the subsequent memoirs of Newton's niece and the correspondence between Johann Bernoulli and Leibniz. The challenge set by Johann Bernoulli may perhaps have been devised to test the power of Newton's infinitesimal calculus. In a letter dated February 1697, Leibniz conjectured to Johann Bernoulli that only the Bernoulli brothers, the Marquis de l'Hôpital, he and Newton were able to solve the problem, since at the time they were the only scientists with the tool required to do so: infinitesimal calculus. This was why, Leibniz explained, when presenting the solutions, Galileo, who was obviously unaware of the calculus, had failed to tackle the problem in his own time.

The challenge inevitably reached Newton, who at that time had already been appointed Warden of the Royal Mint and had retired from scientific activity. Newton received the brachistochrone problem on 29 January 1697. According to his niece, it was handed to him at four o'clock in the afternoon when he returned tired from his work at the Mint. Newton had the solution ready twelve hours later, at four o'clock in the morning. What Newton's niece did not explain, because she was unaware, is that Newton was able to reach back into his memory to identify the cycloid in the brachistochrone problem. According to Whiteside, he must have quickly recognised the same pattern in the problem from the road of revolution offering least resistance to motion in a uniform fluid that he had solved more than a year earlier in *Principia*.

The story does not end here. In Leibniz's exposition of the solutions to the brachistochrone problem he explains his prophecy about those who would be able to solve the problem: "Indeed, it is only proper to note that only those whom I conjectured would be able to solve the problem have been able to do so, and in fact, they are none other than those who have sufficiently penetrated the mysteries of our differential calculus. Hence, in addition to the brother of the author [who proposed the problem] and the Marquis de l'Hôpital in France, I would add [...] Mr Newton." Leibniz chose to omit Fatio de Duillier from the list. Furthermore, from this affirmation, the uninformed reader might assume that Newton was his disciple when it came to calculus.

Fatio attacks... and Leibniz counter-attacks

That however, was more than Fatio was able to take. He prepared his response, published in London in 1699, claiming that "The distinguished Leibniz may venture to ask himself from whom I have learnt the calculus I use. For all intents

and purposes, its general principles and the majority of its rules I have invented myself, from the month of April 1687 and over the following years. At the time, I thought nobody except myself was using this type of calculus. I would not have had lesser knowledge of it had Leibniz not been born. He may be able to claim many followers, but I am not one of them. This fact will remain clear if, in the future, the letters I exchanged with the distinguished Huygens are made public. However, the reality of events has convinced me that Newton has been the first to discover calculus, what are now many years ago. Whether Leibniz, the second inventor, could have taken anything from Newton, is a matter I prefer to leave to the judgement of those who have seen Newton's original letters and manuscripts. Not even Newton's modest silence, nor the constant vanity of Leibniz, attributing the invention of this calculus to himself at every opportunity, shall succeed in deceiving anybody who examines the available material as I have done."

In all likelihood, the old friendship between Fatio and Newton served to further complicate matters: Leibniz may have thought that Newton had convinced Fatio to accuse him of plagiarism, even if Fatio was acting on his own account, seeking to express his gratitude to Newton.

In spite of the directness of the accusation, things were not, at least for the time being, to worsen. Leibniz published a reply to the letter in *Acta eruditorum* where he suggested that Fatio's accusation could have been induced: "I hope he can forgive me if I do not respond to all of his points, until he shows that he was not acting on the initiative of another, and above all of Newton, with whom he has never had the slightest rivalry," and insisting on the independent discovery of the methods of calculus, adding that "whenever the occasion has arisen, I have always been more than happy to proclaim his [Newton's] formidable achievements. He knows this better than anyone else, and has also declared this publicly in 1687 when he stated in his *Principia* that of his new geometric discoveries common to both, neither of the two owed their result to the other but rather to their own reflections, which I had published a decade prior."

Newton's decision to include the two mathematical appendices in his *Opticks*, which was published in 1704, especially *De Quadratura*, was doubtless related to the situation created by Fatio's accusation of plagiarism. It was also influenced by the unchallengeable success Leibniz had enjoyed with respect to his calculus: he had the merit of contributing, together with his followers that soon appeared (Jakob and Johann Bernoulli, and the Marquis de l'Hôpital), to making calculus become a powerful mathematical tool at the grasp of whosoever wished to study it in the last

decade of the 17th century. Alfred Rupert Hall, author of the most complete study on the dispute, notes that "the main divergence between them occurred with respect to the evaluation of the calculus. Was it (like the methods of Huygens and Newton also) simply a continuous development from the methods of analysis known before, a progressive step, or was it a mutation, bringing into existence powerful methods of analysis of a quality totally unlike any that had previously existed? Huygens did not see either Leibniz's calculus or Newton's fluxions as such a mutation. Nor probably did Newton, although he was more conscious, naturally, of the innovation of his own achievements. It is, one may be sure, no small reason in Leibniz's contemporary success and ultimate fame that he did perceive calculus as a mutation, a progressive step so great, like the introduction of algebra, that mathematics would never be the same again."

In a warning at the start of *Opticks*, Newton explained why he added the mathematical opuscules that would disappear in the second edition, published in 1717. "In a letter addressed to Mr Leibniz [...], I made reference to a method thanks to which I had discovered certain general theorems relative to the curvature of curvilinear figures [...] Since a number of years ago I lent a manuscript with such theorems and have subsequently discovered that some of the elements contained therein have been copied, I am taking advantage of this opportunity to publish them."

THE REPLY TO CHEYNE

When Newton remarked in *Opticks* that he had discovered that certain elements had been "copied" he was referring to a book by George Cheyne. However the reference is somewhat unjust because he did not only copy Newton but offered him considerable and sincere praise. "I cannot prevent myself stating that all that has been published by others during the last twenty to thirty years, roughly speaking, regarding these methods or other not dissimilar methods, is merely a repetition or easy corollary of that which Newton long ago communicated to his friends, or the public." Leibniz responded harshly to Cheyne's words, stating that "He makes a clumsy attempt to claim for Newton the method of series of indeterminate coefficients whose determination is obtained comparing their terms, however it was I who published it in 1693, when it was unknown to myself or anybody else, at least in the public domain, that Newton knew of the method." He also remarked that "perhaps Mr Newton discovered certain things before me, as I discovered others before him. I have certainly not found any indication that he knew of differential calculus, or its equivalent, before myself."

The direct accusation of plagiarism made by Newton was not directed at Leibniz, but at the Scottish physician, George Cheyne, who had published a book that brought together and systematised various results on quadratures from Newton, Leibniz, Gregory, Craig and Johann Bernoulli in 1703.

Leibniz prepared a review of *De Quadratura* for *Acta Eruditorum*, anonymously published in the January 1705 volume. Although he would go on to deny that he was author of the review, in the second half of the 19th century Edward G. Guhrauer, one of Leibniz's biographers, discovered the original manuscript of the review with Leibniz's signature. The conflicting paragraph reads as follows: "When a quantity varies continuously, such as, for example, a line varies as it flows through a point that describes it, those momentary increments are referred to as *differences* [...] This has given rise to *differential* calculus and its converse, *summatory* calculus. The elements of this calculus have been published by their inventor, Dr Gottfried Wilhelm Leibniz [...] Instead of Leibniz' differences, Dr Newton employs, and has always employed, fluxions [...] He has made elegant use of these fluxions in his *Principia Mathematica* and other publications, just as Honoré Fabri in his *Synopsis Geometrica* substitutes progressive motion for Cavalieri's method."

As it stands, the previous paragraph does not appear to be written to cause any offence, since the expression "Instead of Leibniz's differences, Dr Newton employs, and has always employed, fluxions", does not presuppose that the differences are prior to fluxions. However, Leibniz made use of a Latin expression that allows a double reading. *adhibet, semperque adhibuit* can be translated as "he employs and has always employed", or as "substituted and has always substituted", in light of which the previous phrase takes on entirely different connotations that are amplified if we bear in mind the references to Fabri and Cavalieri. In the comparison, Newton has the role of Fabri and Leibniz that of Cavalieri. However, what Fabri did was to interpret Cavalieri's indivisibles in terms of flow. Is Leibniz not trying to say to Newton then, that like the other, he had interpreted Leibniz's differences in terms of flow?

It is possible that Leibniz was not conscious of the interpretation that could be given to his words. It is more likely, at least in Hall's opinion, that this was a simple Freudian slip. Leibniz would subsequently maintain that this interpretation of the paragraph was nothing more than an attempt to create a pretext for starting the dispute, he then clarified that whoever wrote the review [*sic*] has used the expression *adhibuit* when discussing Newton, whereas for Fabri the term *substituit* had been used, hence in the case of Newton, the term *adhibuit* would have to be trans-

lated as 'employ', such that the act of using two different verbs to refer to Newton and Fabri precluded a comparison between the two. The polemic had reached extremes where it was necessary to fight word by word.

At first, the review received no response, either because in the first instance the alleged double meaning may have passed unnoticed by Newton and his followers, or more likely because they had not even read it.

The appearance of "Newton's ape"

Nonetheless, the response arrived. It came three years later and was signed by the Scottish mathematician John Keill, who came to become, probably more on his own account than as a result of Newton, a tireless attacker of Leibniz: "A warhorse," as F. Manuel describes him, "of a passion so intense that Newton occasionally had to rein him in." Johann Bernoulli came to refer to Keill as "Newton's ape", and although the word "ape" can also be interpreted as a reference to imitating, we prefer a more literal interpretation as this is doubtless what Bernoulli meant.

JOHN KEILL (1671–1721)

John Keill was born in Edinburgh in December 1671, where he was a student of David Gregory. He belonged to the first generation of those educated in Newton's recently published philosophy. He graduated from Oxford, where he moved with Gregory to take up his appointment as Savilian chair. Keill occupied the position from 1712 until his death in August 1721. He joined the Royal Society in 1700. Keill was a fierce proponent of Newton's philosophy, standing by his side in a distinguished and occasionally strident manner in a number of controversies.

At the start of 1709, Keill accused Leibniz of plagiarism in the *Philosophical Transactions*: "All of these [propositions] follow from the now highly celebrated Arithmetic of Fluxions which Mr. Newton, beyond all doubt, First Invented, as anyone who reads his letters published by Wallis can easily determine, the same Arithmetic under a different name and using a different notation was later published in the *Acta Eruditorum*, however, by Mr. Leibniz."

Philosophical Transactions was a journal published by the Royal Society, which lent a more official appearance to the accusation. Leibniz was, as a consequence of his first visit to London in 1673, a member of the Society, and in 1711 he requested the rectification of Keill's accusation. Perhaps Leibniz failed to appreciate that the dispute over infinitesimal calculus was itself affected by the criticisms against the metaphysical foundation of Newtonian gravity, criticisms originating from Leibniz's camp, and even from Leibniz himself. Unfortunately nor did he sufficiently appreciate, when he went to seek support and defence from a Royal Society of which Newton was then president, the nationalistic connotations that the dispute, together with the criticisms of gravity, had assumed, becoming an attack by the continental scientists on British science as represented by its greatest figure, Newton. These nationalistic references, both direct and indirect, can be found everywhere in the many letters and documents published by those involved in the dispute. Consider William Jones for example, who informed Roger Cotes in 1711, "I have little news to tell you, just that the Germans and the French have violently attacked the philosophy of Sir Isaac Newton [...]." Or Leibniz, when he wrote to Johann Bernoulli in 1713, that "for many years, the

Engraving showing a session of the Royal Society presided over by Newton at its premises in Crane Court.

English have been so swollen with vanity, even the most distinguished of them, that they have not wasted the opportunity to snatch things from the Germans and claim them as their own [...]. Now who seeks to deprive Nicholas Mercator and Holstein of the glory of having been the first to discover a series and are disgusted with me for having defended the honour of a very good man, my friend."

Instead of the letter of rectification requested from Keill, Leibniz received a very different response: more accusations lodged by Keill in a new letter read at the Royal Society in a session presided over by Newton and held on 24 May 1711. In it, Keill attacked Leibniz: "Surely Leibniz's merits in the field of knowledge are considerable, I acknowledge this without problems, just as nobody who has read his contributions can deny that Leibniz is an expert in the most incomprehensible parts of mathematics. Given that he possesses such indisputable riches, I certainly do not understand how he can wish to satisfy himself with the spoils stolen from others [...] Hence, as this distinguished figure has appealed to the Royal Society and wishes me to publicly testify that I had no intention of slandering him, I must show, to rid myself of the accusation of slander, that Mr. Newton was the true and first discoverer of the arithmetic of fluxions or differential calculus, and it was after he had sent clear and obvious indications of this method to Leibniz that it became easy for him to arrive at the same method."

Leibniz's fate was in the hands of the Royal Society

When Leibniz received Keill's letter, he responded by acknowledging the joint paternity of calculus: "There is no reason why I must instruct, refuting your reconstruction of my method of discovering the knowledge of things, one who [Keill] has such insufficient experience as to be able to judge the methods of discovery; my friends know that I proceeded on quite a different basis and following quite different objectives. It is useless on his part to appeal to the Leipzig *Acta* to allow him to excuse his own words, because therein I do not find anything that detracts anything from anyone, on the contrary, I find that in paragraphs here and there, each receives their dues. I and my friends have also declared on a number of occasions, that the distinguished discoverer of fluxions arrived there on his own account and based on basic principles like our own. I have no less right than he [Newton] to claim the discovery," and seeking the Royal Society's protection from Keill's insults, adds "I submit to your sense of justice, to determine if such vacuous and unjust braying, of

which indeed I think Newton himself would disapprove, being a distinguished figure who knows how events happened, should be silenced; I hope that he will freely give his opinion on the matter." Here it is fitting to mention Newton's thoughts on second inventors "Second inventors have no right; the sole right is in the first inventor, till another finds out the same thing a-part, in which case to take away the right of the first inventor, and divide it between him and that other, would be an act of injustice." Moreover, "Second inventors, even being an inventor as such, are not entitled to any honour, their title or right is nil Thus, what is to be said of those who do not even possess certain arguments to show that they are second inventors?"

Leibniz's fate was written. To cool his wrath, "Newton stumbled upon an unexpected solution," writes Westfall, "Leibniz himself had appealed to the justice of the Royal Society Very well, then let the Royal Society judge the matter" A commission was nominated for this purpose, composed of Newton's friends and defenders and, to give the process some semblance of impartiality, the representative of the Prussian Kingdom in London was also invited, joining the commission a week before the verdict and, as such, being unable to participate in its preparation The composition of the commission was kept secret and not made public until the second half of the 19th century.

It only took fifty days for the commission to review the documents and deliver its verdict Perhaps this was actually quite a long time, given that Newton almost wrote the work in its entirety It contained four points and, although not directly accusing Leibniz of plagiarism, it sowed sufficient doubts so as to make it possible to draw that conclusion. The final point noted "that the differential method is one and the same with the method of fluxions, excepting the name and the mode of notation, Mr Leibnitz calling those quantities differences, which Mr Newton calls moments or fluxions, and marking them with the letter *d*, a mark not used by Mr. Newton And therefore we take the proper question to be, not who invented this or that method, but who was the first inventor of the method, and we believe, that those who have reputed Mr Leibnitz the first inventor, knew little or nothing of his correspondence with Mr Collins and Mr Oldenburg long before; nor of Mr. Newton's having that method above fifteen years before Mr. Leibnitz began to publish it in the *Acta Eruditorum* of Leipsic For which reasons we reckon Mr Newton the first inventor, and are of opinion that Mr. Keill, in asserting the same, has been noways injurious to Mr. Leibnitz."

The committee's verdict was accompanied by the documents and letters upon which it had been based (conveniently annotated in Newton's interests) and all this was published under the title *Commercium Epistolicum D Johannis Collins, et Aliorum de Analysi Promota* by the Royal Society in 1712. Few copies were made (the document was not made available for sale) and the distribution was discretionary but highly specific. Newton rectified this scarcity of copies in 1722, six years after Leibniz' death, with the publication of a second, extended edition.

A preface was added to the *Commercium Epistolicum*, written by Newton in order to "suitably orientate" the reader in the dispute.

"If it had acted justly, I would have been informed that the Society wished to examine the matter in depth," Leibniz complained a year and a half later regarding the partial nature of the verdict, "and I would have had to have been given the opportunity to state if I would set out my arguments and if I considered any of the judges to be dubious. Hence it had been decided after having heard only one side of the tale, such that the nullity of the procedure is clear."

The documents contained in the *Commercium Epistolicum* were, for a number of reasons, ill suited to resolving the polemic. However they left Leibniz in an

ONE DISPUTE, TWO ATTITUDES

Newton also maintained a somewhat dramatic attitude, which, as F. Manuel notes, is rooted in his traumatic separation from his mother. "When confronted with the possibility that what was his could be taken from him, he reacted with a violence proportional to the fear and rage created by this first serious dispossession."

Leibniz also took the dispute very seriously, but in a different way, even allowing himself to joke about it from time to time. "It is impossible to provide full information without going on, and it is impossible to avoid the judges yawning from time to time should they have to deal with trials as long and large as ours. But [...] you could be like the cobbler from Leyden [...]. When there were disputes about theses at the university he always attended the public sessions. Finally someone who knew him asked if he understood Latin. 'No,' he replied, 'and I shall not even take the trouble to learn it.' 'Why then do you come to this auditorium so much when only Latin is spoken?' 'Because I enjoy judging the debates.' 'And how can you judge if you do not even understand what is said?' 'Because I have another way of judging who is right.' 'How?' 'When I see the face of someone getting angry and red, I know that they are lacking reason.'"

extremely complicated situation, to such extent that for Johann Bernoulli it seemed they may even have been falsified. Thus, Leibniz was informed in 1714 that "as some of the letters displayed in *Commercium Epistolicum* appear to me highly dubious; if they have not been completely fabricated, then they have at least been altered and falsified."

Until the second half of the 19th century, when Gerhardt's edition of Leibniz' mathematical writings was published, there was no documentary evidence to attest to Leibniz having made his discovery independently of Newton. In some senses, *Commercium Epistolicum* and Newton's mathematical works, published belatedly during the first decades of the 18th century, decided the battle in favour of Newton, at least in terms of documents. However, the mathematical battle would be won by Leibniz, thanks to his successors. First the Bernoulli brothers and later on Euler would ensure that his differential calculus won out as the 18th century passed, such that by the start of the 19th, even the British were forced to accept the greater development achieved on the continent.

Charta Volans and the "leading mathematician"

Leibniz responded to *Commercium Epistolicum* almost a year later, with a defamatory anonymous letter against Newton in the famous *Charta Volans*. In it, he accused him of plagiarism, specifically of having developed his method of fluxions based on what Newton had set out in his letters in response to the *Epistola prior* and *posterior*.

Leibniz decided on his strategy of argument for *Charta Volans* after receiving a letter from Johann Bernoulli, informing him of the appearance of the *Commercium* and accusing Newton of plagiarism. Leibniz published extracts of Bernoulli's letter in *Charta Volans*, attributing it to an anonymous but "leading mathematician".

Johann Bernoulli played an important role in the dispute and his conduct was highly dishonourable. While he encouraged the confrontation between Leibniz and his rival, he feigned neutrality in letters to Newton and his friends in order to avoid attracting his enmity and wrath. When Leibniz included the extract of Bernoulli's letter in the *Charta Volans* (in spite of the fact he asked to remain on the margins of the conflict with Newton and his people) he discovered his trick. Leibniz's indiscretion, unconscious or intentional, made Bernoulli's conduct even clearer when he published another letter in December 1715 in which he explicitly stated just who was the "leading mathematician" of the *Charta Volans*.

The sequence of letters exchanged with Newton in 1719 provides a good account of Johann Bernoulli's attitude during the polemic and his subsequent explanation of the "leading mathematician" letter that Leibniz included in the *Charta Volans*. In one he claimed, "I don't know how it happened, but after the start of the fierce dispute that took place some years ago between the geometers of the British and German nations (much to the embarrassment of mathematical science), I, who am neither British nor German, nor even Swiss, and so averse to taking sides, I who would seldom do anything to voluntarily involve myself in the disputes of others, at any rate ceased to be in your favour, as the rumours claim [. . .]. Hence, I have no doubt, my great Sir, that there have been many lies and interventions regarding myself that may have at least weakened if not totally destroyed, the favourable opinion you previously held of me [. . .] I have spoken with particular praise of you and your discoveries when the occasion has arisen: what more can one who considers your merits to be of the greatest grandeur do? Those who have heard me or read my works will be able to confirm that I have praised and exalted them as marvellous on every occasion and everywhere, in my letters, lectures, classes and conversations [. . .] Moreover, beyond all doubt, those who have informed you that I was the author of one of the *Charta Volans* in which your name is perhaps not mentioned in a sufficiently honourable way, are unquestionably mistaken. I passionately entreat you, famous Sir, and implore you in the name of that most sacred to humanity, that you are fully convinced whatever has been anonymously published in this manner has been falsely imputed to me. Because it has not been my custom to anonymously publish that which I neither wish nor dare to recognise as my own."

In December 1719, perhaps thinking that someone could have access to the documents of the late Leibniz and copy the letter reproduced by him in *Charta Volans*, where Bernoulli effectively referred to Newton dishonourably, he wrote once again to Newton with a gall that can scarcely be equalled: "I don't remember having written to Mr Leibniz that day, although nor can I deny it, since I do not keep copies of all the letters I have written. However, if, perhaps, among the innumerable letters that I have written, one was found with the exact day and year mentioned, I would have no objection to asserting in full confidence that nothing contained therein could in any way weaken your reputation for honesty. Furthermore, I never gave permission [to Leibniz], indeed certain letters, particularly those that would displease you, were published against my trust and wishes."

The lion sharpens its claws

Newton came to copy out the *Charta Volans* in his own hand, as if writing the insults directed at him by Leibniz would transmit some kind of energy to help feed his cravings for revenge. Newton intended to respond to the publication of the *Charta Volans* personally, and in 1714 he prepared a reply, although this was never sent; instead he preferred to encourage Keill.

Newton's response to the *Charta Volans* was the *Account*, written in the form of an anonymous review of the *Commercium Epistolicum* and published in English in 1715 in *Philosophical Transactions*. Various manuscripts of the *Account* in Newton's writing have been conserved with versions and variations broadly in line with the text that was finally published. It is a good example of the obsession that gripped Newton in waves during the worst years of the dispute (1712–1716) and even for some years after Leibniz's death. The English scientist appears to be seized by a compulsive desire to set out his version of events to the world, to argue over and over against Leibniz based on the letters exchanged between them and others. Newton gave himself over to writing letters, memorandums and observations, which he would re-write repeatedly, polishing a phrase here, modifying an argument there, changing a quotation here, honing an insult there. Furthermore, all, or almost all, of these writings (versions of versions of versions of letters that were never sent or badly published in addenda to other texts after his opponent's death) are conserved as mute testaments to the fury that seized him as he wrote them.

The *Account* claims to be anonymous, although few could doubt its authorship at the time: it is a long, brutal and defamatory report against Leibniz, in which Newton, who did not believe the documents contained in the *Commercium Epistolicum* were able to explain the story on their own, gave them a voice, interpreting them in his own interest and duly pointing out the consequences, returning all the accusations of Leibniz and his friends on a larger scale.

In the last three pages of the *Account*, Newton refers above all to the criticisms of the nature of gravity, which, according to Leibniz, he had transformed into a dark and miraculous property of the same Aristotelian type that the medieval scholastics believed explained the motion of bodies. Newton was conscious of not having explained the cause of gravity and was dissatisfied that it was a property that acted over a distance, even in a vacuum, but nonetheless did not wish to renounce a theory of such great predictive value. This is explained in the general note he added at the end of the second edition of *Principia*: "However, I have still been unable to deduce, based on phenomena, the reason behind these properties of gravity and I

GOD IS NOT A CLOCKMAKER

As an interesting aside to the dispute over calculus, between November 1715 and October 1716 (Leibniz died in 1716), there was an interesting philosophical controversy between Samuel Clarke and Leibniz on metaphysics and natural philosophy. The origins of the polemic lay in Leibniz's criticisms of the metaphysics of gravity and the role of God within Newton's natural philosophy. Newton came to Clarke's aid in the polemic (although without coming to address him as he did with Keill but he always complained that Leibniz wished to avoid the question of the discovery of calculus in favour of a new dispute on metaphysics, where the German was confident of being able to manoeuvre with greater ease, having once observed that Newton had little success in metaphysics). Leibniz's criticisms mainly focused on certain paragraphs from *Opticks* in which he appeared to deduce that God must intervene in the universe from time to time to avoid the stars falling on top of each other or the Solar System becoming confused. Just as a clockmaker is required to adjust his clock from time to time. Leibniz vehemently attacked this: "Mr Newton and his followers have a very strange opinion when it comes to the work of God. According to them, God is required to adjust his clock from time to time. Otherwise it would cease to move. This machine of God's is also so imperfect that he is obliged to adjust it once in a while by means of an extraordinary aid, and even repairing it, as a clockmaker would do with his work."

have not been able to devise a hypothesis. That which is not deduced from phenomena must be referred to as a hypothesis, either metaphysical or physical, or with dark, or mechanical properties, and has no place within experimental philosophy [...] And it is quite clear that gravity exists in fact and acts according to the laws set out by us and is sufficient for all motion of the heavenly body and our sea."

Leibniz tried to respond to the *Account* by preparing his own version of events, entitled *Historia et Origo Calculi Differentialis*, although he soon came to see the impossibility of composing a piece as complete, detailed and abundant in letters and documents, as Newton's *Commercium*. He acknowledged this in his peculiarly distant style of writing in the first pages of *Historia et Origo*: "Being away from home when these matters were divulged by his adversaries, returning after a period of two years and being distracted by other business, he was unable to recover or consult the rest of his old *comercio epistolar* in which he would have been able to instruct himself of acts performed in such distant times, more than forty years past, since he had not conserved copies of letters and his other writings from years gone by."

It should be noted that it was much harder for Leibniz than Newton to write an article so thoroughly grounded and supported by old documents, among other things as a result of the incredible volume of letters written or received by Leibniz throughout his life, of an order of magnitude greater than those of Newton. In this respect, Leibniz's excuse about being unable to locate his letters is certainly credible. "To respond point by point to the work published against me [...] would oblige me to search for my old letters, some of which have been lost; moreover, on the majority of occasions, I have not kept the drafts of my own. Others lie buried under a large mountain of papers that could not be ordered without considerable time and patience."

As such *Historia et Origo* is more the product of Leibniz's memory than the *Account*. Perhaps Leibniz had thought of expanding and completing it, but the fact is that his death, which occurred in November 1716, left it in its current form. *Historia et Origo* remained unpublished during the dispute and in fact, a whole 130 years would pass until it saw the light of day in 1846.

Statue of Abbé Conti erected in 1781 in Padua, his hometown.

When Leibniz's patron became Newton's king

John Chamberlyne and Abbé Conti led two interesting but nonetheless unfruitful attempts at reconciliation that took place between Leibniz and Newton in 1714 and 1715. Both figures came from different backgrounds. Chamberlyne was a member of the Royal Society, as well as being a minor political figure. He had exchanged letters with Leibniz since 1710 and was on favourable terms with Newton. For his part, Antonio Schinella Conti was a cleric from Padua, who came to England to observe a solar eclipse and remained there for a few years. He was on favourable terms with Leibniz and his charming personality favoured the start of a friendship with Newton. According to F. Manuel: "Conti was one of those os-

tentatious and crafty figures from the 18th century, a poetaster, [illegible] Racine and Pope, enthusiast of the sciences, and an amateur interested [illegible] in princesses and natural philosophers in England, France, Germany [illegible] was interesting about their initiatives was the political element presented [illegible]

One the one hand, Chamberlyne was concerned about Leibniz's [illegible] councillor of the Hanoverian dynasty that was soon to take the Engl[illegible] made this implicitly known to Newton in May 1714: "I am extremely s[illegible] having been able to wait for you this afternoon to consider the letter se[illegible] Mr Leibniz regarding you; [...] I am happy that you make use of it as [illegible] although with the prudence that, in your best understanding, you see advis[illegible] proportional for a gentlemen who, in the case of Leibniz, is held in the hig[illegible] teem by the Court of Hanover." However, Chamberlyne's intervention [illegible] very nearly did) extremely badly and cost him his relationship with New[illegible] Westfall explains "Blessed are the councillors because they will inherit the en[illegible] of both parties."

At the start, the coronation of Leibniz's patron, George I, as king of Great Br[illegible]ain and Ireland could not help but work to his advantage in the dispute "I can easily believe," wrote Johann Bernoulli, "that once his most Serene Prince ascends to the throne of Great Britain, the Royal Society will not wish its opinion to be [illegible] already published in the *Commercium epistolicum* [...] under its own authority and its own name Perhaps not even Keill would have wished his French pamphlet to have been made public if he had intuited the welcome changes to take place in Britain's affairs shortly after."

Keill himself, in a letter to Newton, dated November 1714, a few months after the death of Queen Anne, expressed certain concern about the possibility of Leibniz travelling to England with the new king: "I hope that Mr Leibniz [illegible] refrains from the imprudence of showing his face in England. Should he do so, I am convinced he will not meet with many friends." Leibniz was also hoping for royal support in the dispute because, although his relationship with his patron had always been difficult, and indeed at times terrible, he maintained excellent relationships with Sophie, his mother, and Caroline, his daughter-in-law (and Princess of Wales upon George's coronation). However, none of this happened and it would appear that the king showed no interest in prejudicing Newton in the dispute with Leibniz.

However, even if only tangentially, Abbe Conti did succeed in implicating the new King of England, George I, and a group of Hanoverians from his court in

London. Together with other eminent figures, they had examined the documentation provided by Newton regarding the dispute at a meeting, after which they decided that, not being sufficient, Newton should write his version in a letter to Leibniz. The letter should be approved by the king and Leibniz would be required to respond with his version of the story. With this Newton and Leibniz began a final exchange of letters that grew increasingly long and acrimonious, until the new attempt at reconciliation threatened to inflame the dispute even further. Leibniz's death, on 14 November 1716, put an end to any possibility of future resolution. Newton, in an act that displays his personality's most vindictive streak, published this final exchange of letters after Leibniz's death, together with some severe *Observations* on Leibniz's final letter.

"M. Leibniz est mort, et la dispute est finie," wrote Abbé Conti to Newton on 10 December 1716. But this was not the case: the disappearance of one of the contenders failed to end the dispute, since the main champions on both sides, John Keill ("Newton's ape") and Johann Bernoulli (the "leading mathematician" and Leibniz's friend and student), remained alive (Keill until 1721 and Bernoulli until 1748). And Newton too was to live another ten years, the last six with the dispute as his prime concern. During those years, he continued to fill more and more sheets arguing his rights to the invention of calculus, about what was proved by the old documents and letters, about his contempt for those who sought to divest him of his discoveries or criticised his science. These pages remained largely unpublished, like so many others Newton dedicated to mathematics, alchemy, theology and history. He even came to write in his own hand a list of observations bitterly questioning and censuring part of the *Eloge* on Leibniz published by Christian Wolff in *Acta eruditorum* after his death, and the *Eloge* of Leibniz written by Fontenelle secretary of the Académie Royale des Sciences in Paris.

"The morality inherent in the puritan world," writes F. Manuel, "like all Christian morality, prescribes love of God and one's fellow men, the two principles to which Newton reduced all religion. However puritanism also ordered the eradication of evil. To love and destroy – an ambivalent command."

Chapter 6

The Infinitesimal Tamed

Infinitesimal calculus did not have an easy childhood. From the end of the 17th century, when Newton and Leibniz had introduced the concept, through to its subsequent development and perfection during the 18th century, calculus was 'contaminated' by infinitely small quantities (infinitesimals), as well as infinitely large ones. An infinitesimal quantity is an indefinitely small quantity that does not reach zero. The fact that it is non-zero allows it to appear on the denominator of coefficients, and since it is infinitely small, it can be taken to be zero when we wish to simplify expressions. On the other hand, an infinitely large quantity is effectively invariant when added to a normal number, meaning that if N is an infinitely large number, we have the following disconcerting inequality: $N + 1 = N$.

Naturally, with these peculiar properties, the real existence of these infinitely small or large quantities was doubted on numerous occasions. Infinitesimal calculus was often attacked for being based on infinitesimal quantities. The question was, how is it possible to arrive at correct results from a method based on such a logically cloudy concept?

The mathematicians who began using infinitesimal arguments in the 17th century (including Kepler, Cavalieri, Fermat, Wallis, Pascal and Barrow), appealed on a number of occasions that they had not done anything other than what Archimedes had already done, although without going to the trouble of ratifying their results with rigorous proofs, as the Greek had done. At that time, the known works of Archimedes had been published in the second half of the 16th century, and almost half a century had passed in which mathematicians had been digesting Archimedes' difficult and sophisticated techniques. However, once this had happened, Archimedes became the most heavily cited author for the whole of the 17th century, throughout which, as we have seen in Chapter 2, mathematicians were divesting their clothing of Greek geometric methods and adopting the arithmetic and algebraic methods that mathematics began to acquire as the 17th century advanced (above all

after the discovery of Descartes' and Fermat's analytic geometry). At that point, mathematicians were more concerned about the discoveries they could make using the strange properties of infinitesimals and preferred not to spend any time or effort cladding them with the corresponding geometric proofs.

On many occasions the reason advanced to justify this removal of the burden of proof was simply avoiding work. "All this could be proved using Archimedes' techniques but this would require a great deal of effort," wrote Cavalieri in 1635.

Newton, Leibniz and infinitesimals

Nor did the founders of infinitesimal calculus provide a suitable justification of their methods. Both Newton and Leibniz were conscious of the logical weakness of their methods and each attempted in their own way if not to resolve them, at least to ameliorate the problem.

Hence Newton tried to avoid the use of infinitesimal quantities by using a limit step process, and while his attempts were not particularly successful, his efforts did, however, serve as an inspiration for Cauchy's logical foundations. For example, Newton sought to justify his understanding of the quotient $\frac{0}{0}$ obtained when h is equal to 0 in the quotient of increments

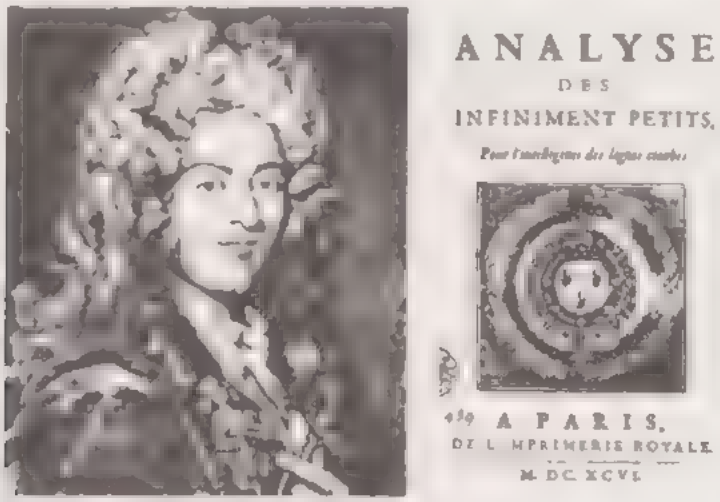
$$\frac{f(x+h)-f(x)}{h},$$

used to define the derivative $f'(x)$ of the function f at point x (note that Newton would not have used the terms derivatives or functions, and nor would he have used this notation). To do so he outlined the concept of a "vanishing quantity" (the difference $f(x+h)-f(x)$ or the number h itself would be vanishing quantities that "vanish" when h is 0) and "last proportions of vanishing quantities" (the value of the quotient prior to setting h to 0). It seems perfectly clear that Newton is referring to a limit step process when he talks of the "last proportions of vanishing quantities" in order to justify the $\frac{0}{0}$ indeterminacy to which the previous quotient is reduced when we let h equal 0, although we never manage to suitably define it. Newton himself aware of the problem, used physical comparisons to elucidate the concept. "It could perhaps be objected that there is no last proportion of vanishing quantities, because before the quantities have disappeared, the proportion is not the last, and

when they have disappeared, there is no proportion. However the same argument can be used when a body reaches a certain position and stops: it has no last speed, because before reaching that position, the speed will not be its last speed and once there it will be zero. However the answer is easy: we understand the last speed to be that at which the body is moving in the same instant as arriving, not before or after, that is to say the speed at which the body arrives at its last position, and that at which it stops moving. In the same way, the last proportion of vanishing quantities must be understood as the proportion of quantities neither before they disappear nor after they have disappeared, but when they disappear."

Infinitesimal quantities were more essential to Leibniz's calculus than to Newton's. For example, they appeared as an inherent part of Leibniz's concept of curves. For Newton they were generated by a point in motion: "I believe mathematical quantities are not constituted of extremely small parts but are described by a continuous motion. Curves are described and generated in this way, not by a layout of parts, but by the continuous movement of points." For Leibniz they were made up of straight line segments of infinitesimal length: "To find the tangent, trace the straight line that joins two points of a curve at an infinitely small distance, or the extended side of a polygon with infinite angles which is for us equivalent to the curve," wrote Leibniz in 1684.

And this concept of a curve appeared even more clearly in the Marquis de l'Hopital's book *Analyse des Infiniment Petits pour l'intelligence des lignes courbes* (1696), whose second postulate read: "Assume that a curve can be considered as the aggregate of an infinite number of infinitely small lines, or (the same) as a polygon



The Marquis de l'Hopital's Analyse des infiniment Petits pour l'intelligence des Lignes Courbes was the first textbook on Leibniz's version of infinitesimal calculus

with an infinite number of sides, each with an infinitesimal length, where the curve of the line is determined by the angles between these sides."

Leibniz justified his infinitesimal arguments in a similar way to other colleagues who preceded him: "One takes quantities as large or small as necessary to ensure that the error will be less than the given error, such that one differs from Archimedes' style only in the expression, our method being more in line with the art of invention." Therein lies an important point: at that time, he was more interested in discovering than proving.

"The ghosts of departed quantities"

In spite of the spectacular advance Newton and Leibniz's infinitesimal calculus represented for science and mathematics, the numerous criticisms about the shakiness of its foundations were justified.

Without a doubt, the most severe criticism was published in England by bishop and philosopher George Berkeley. In 1734, he published a book entitled *The Analyst*, which reviewed the ideas of calculus in order to demonstrate their lack of logical foundation.

EDMUND HALLEY: THE UNBELIEVER

Berkeley's book carried a subtitle: *A discourse addressed to an infidel mathematician*. The infidel mathematician must have been Edmund Halley, also an astronomer, who was always famous for his atheism and who, on one occasion, had caused a sick man to decline the attention of Bishop Berkeley, convincing him of the lack of solidity of the Christian doctrine. With his book, Berkeley sought to demonstrate that the arguments of infinitesimal calculus lacked solidity, just as those of religion. The subtitle of the book continued: "Wherein it is examined whether the Object, Principles, and Inferences of the Modern Analysis are More Distinctly Conceived, or More Evidently Deduced, than Religious Mysteries and Points of Faith." It then went on to state: "First cast out the beam out of thine own Eye, and then shalt thou see clearly to cast out the mote out of thy brother's eye."

Berkeley's book also outlined a series of questions to be considered: "Query 62: Whether Mysteries may not with better right be allowed of in Divine Faith than in Humane Science? Query 63: Whether such Mathematicians as cry out against Mysteries have ever examined their own Principles?"

For instance, Berkeley denounced the way in which Newton calculated the formula for the derivative of a product in *Principia* (see Chapter 3) as being an example of trickery. After recalling Newton's proof, Berkeley wrote that "It is plain that the direct and true Method to obtain the Moment or Increment of the Rectangle *AB* is to take the Sides as increased by their whole Increments and so multiply them together, $A + a$ by $B + b$, the Product whereof $AB + aB + bA + ab$ is the augmented Rectangle, whence if we subduct AB, the Remainder $aB + bA + ab$ will be the true Increment of the Rectangle, exceeding that which was obtained by the former illegitimate and indirect Method by the Quantity ab. And this holds universally be the Quantities a and b what they will, big or little, **Finite or Infinitesimal, Increments, Moments or Velocities.**"

Referring to the method of calculating fluxions using vanishing increments, he observed that "it must, indeed, be acknowledged that he used Fluxions, like the Scaffold of a building, as things to be laid aside or got rid of, as soon as finite Lines were found proportional to them. But then these finite Exponents are found by the help of Fluxions. Whatever therefore is got by such Exponents and Proportions is to be ascribed to Fluxions: which must therefore be previously understood. And what are these Fluxions? The Velocities of evanescent Increments? And what are these same evanescent Increments? They are neither finite Quantities nor Quantities infinitely small, nor yet nothing. May we not call them the Ghosts of departed Quantities?"

Euler and the analysis of infinities

If Newton and Leibniz are considered the founders of calculus, Leonhard Euler is considered likewise for mathematical analysis, a branch of mathematics that covers the infinitesimal methods of differential and integral calculus. In this respect, his books *Introductio in Analysin Infinitorum* (1748), *Institutiones Calculi Differentialis* (1755) and *Institutiones Calculi Integralis* (1768–1770) are key texts that provided a **suitable structure for the new discipline.**

Introductio in Analysin Infinitorum represents the birth of mathematical analysis; it was for analysis what Euclid's *Elementos* was for Greek geometry. In this work, Euler situates the function as an object of study for analysis, systematising the work of his predecessors on elementary functions, studying them without making reference to differential or integral calculus, although making an almost obscene use of infinitely large and small quantities (see the appendix for more information) and

avoiding, as far as possible, geometric arguments (drawings) in favour of analytical ones (formulae). The structuring of differential calculus is set out in the second work of his trilogy.

In spite of the fact that Euler was a scientific descendent of Leibniz (grandson, we could say), in *Institutiones Calculi Differentialis* he returns to the differential in the sense of difference, although introducing a change to Leibniz's calculus that approximates the interpretation of Newton's "vanishing increments".

Faced with the perennial doubt surrounding infinitesimals, Euler thought that what really mattered was not knowing what they were but how they behaved. In this respect, for Euler, infinitesimal quantities were zero, or ended up being zero, what mattered was that these quantities were subject to having quotients with each other. These quotients, which in principle are $\frac{0}{0}$, can represent a perfectly determined finite quantity. Hence, the differentials dx, dy are important insofar as they define a quotient $\frac{dy}{dx}$ that can be determined. Calculus is precisely a way of determining when these increments vanish. "A method to determine the proportion of the vanishing increments, which the functions take when the variable of the

EULER 'THE GREAT'

Euler is one of the greatest mathematicians of all time and [illegible] of the 18th century. Born in Basel in 1707, [illegible] received private classes from Johann Bernoulli (remember Bernoulli was one of Leibniz's followers).

In 1727 he joined the St Petersburg Academy of Sciences, where he worked until he moved to the Berlin Academy in 1741. In spite of disagreements with King Frederick the Great of Prussia, he remained in Berlin for twenty-four years, after which he returned to the Academy of St Petersburg, where he died in 1783.

Leonhard Euler in a portrait by Johann Georg Bruckner.

function is modified by one of these increments," he wrote in *Institutiones Calculi Differentialis*. That is to say, in Euler's analysis, the coefficient of increments

$$\frac{f(x+\Delta x)-f(x)}{\Delta x}$$

begins to appear, giving rise to the derivative of a function, a concept that would come to replace the differentials dx, dy that held such an important position in Leibniz's calculus. This change, introduced by Euler, brought the concepts of Leibniz's differential calculus to the idea of the limit that would later be used by Cauchy to provide its foundations.

The last work in Euler's trilogy, *Institutiones Calculi Integralis*, deals, as its name suggests, with integral calculus. This is explicitly understood as the inverse operation to differentiation. Despite still being geometrically related to area, integration had lost the autonomous property defended by Leibniz and which Cauchy helped to recover with his construction of the definite integral.

D'Alembert, Lagrange and Karl Marx

Over on the continent, as the 18th century wore on, a figure of much higher mathematical standing than Berkeley became highly critical of infinitesimal quantities. Thus, D'Alembert wrote that "a quantity is something or nothing; if it is something, it has not yet vanished; if it is nothing, it has literally vanished. The supposition that there is an intermediate state between these two is a chimera."

In *Encyclopedie*, D'Alembert provides a primitive definition of a limit that would be used by Cauchy as his point of departure for the foundations of calculus: "One says that the quantity is the limit of another quantity, if the second approaches the first closer than any given quantity, however small." In his article *Differentiales*, in the same *Encyclopedie*, he indicated how to provide a rigorous foundation of calculus: "Newton used a different principle and we can say that the metaphysics of this great mathematician regarding the calculus of fluxions is highly precise and illuminating, in spite of the fact that it gives us an imperfect view of his thoughts. He never considered differential calculus as the study of infinitely small quantities, but as the method of first and last ratios, or in other words, the method for finding the limits of ratios. One feels that the admission of infinitely small quantities is only used to abbreviate and simplify arguments, but differential calculus need not suppose the

Marble sculpture of the the French mathematician and philosopher D'Alembert

Karl Marx showed great interest in the foundations of calculus

existence of such quantities. Moreover, this calculus merely consists of the algebraic determination of the limit of a ratio."

Lagrange pursued a different line of argument and in his book *Théorie des Fonctions Analytiques,* published in 1797, he develops the idea of defining the derivative $f'(x)$ of a function f at a point x as the coefficient of h in the expansion of the power series of h of the function $f(x+h)$. It is precisely to Lagrange that we owe the term 'derivative' and the primed notation (f') to denote the derivative of a function f. However, his attempt was unsuccessful and doomed to fail since, as Cauchy would later explain, a function f has no reason to coincide with the power series obtained from it.

As an aside, is should be noted that Lagrange's efforts to provide the foundations of calculus were held in high esteem by Karl Marx, the founding philosopher of Marxism. Although belatedly (1863–1883), when Weierstrass' work had already provided solid bases for analysis, Marx even came to write a number of works on the concept of the derivative and the differential. He identified three periods in the evolution of calculus: Leibniz's and Newton's mystic differential calculus; the rational differential calculus of D'Alembert; and the algebraic puritanism of Lagrange. Of the mathematicians from the start of this

period, he noted that "mathematicians really believed in the mysterious character of the newly-discovered means of calculation which led to the correct and, particularly in the geometric application, surprising result by means of a positively false mathematical procedure." However, he was more understanding with D'Alembert and Lagrange "D'Alembert stripped differential calculus of its mystic veil, and took a major step forward [...] Lagrange took the Taylor theorem, which is more general and extensive, and at the same time, in operational formula of differential calculus, as a point of departure."

Cauchy

In the first half of the 19th century the definitive rigorous foundations of infinitesimal calculus were laid down. The task was essentially begun by Cauchy and completed by Weierstrass (Bernhard Bolzano's contributions on continuous functions were also important, although they lie beyond the scope of this book.)

Cauchy succeeded in generating a school of mathematical thought concerned with increasing the degree of rigour that had previously been required from proofs, a school that would mark the development of mathematics in the 19th century.

The classes given by Cauchy (much to the exasperation of his students) from 1817 to 1830 at the École Polytechnique and above all the texts he published, are clear examples of his efforts. From the point of view of our history, the most significant of these are *Cours d'Analyse* (1821) and *Resumé des Leçons sur le Calcul Infinitésimal* (1823).

Cours d'Analyse was Cauchy's response to the criticism he received from his colleagues on the senate regarding his method of teaching at the École in delivering his course on analysis and mechanics for first year students. The introduction clearly sets out his aim: "I have attempted to develop the methods required by geometry without making any recourse to arguments taken from the generality of algebra. While arguments of this sort are sometimes permitted, above all when passing from convergent to divergent series and real to imaginary quantities, they cannot be considered as anything other than an indication of the path to the truth and not representative of the exactitude of which the science of mathematics should take pride." This 'generality of algebra" that Cauchy mentions refers to the fact that from the end of the 16th century it was commonly held that what was true for the real numbers was also true for complex numbers. Indeed, what was

true for finite magnitudes could also be extended to infinitesimals, and what was true for convergent series also held for divergent ones.

As a basis for the foundation of infinitesimal calculus, Cauchy proposed the concept of the limit, which he defined thus "When the successive values attributed to a variable indefinitely approach a fixed value, such that in the end they differ from it as much as one wishes, this final quantity is the limit of all the others."

Using the concept of the limit, Cauchy defined infinitesimals as variables that tend to zero: "When the successive values of a variable decrease indefinitely in such a way that they eventually become less than any other given quantity, this variable is called an infinitesimal. The limit of this variable is 0."

Similarly, for handling series, he proposed the definition of the limit of a sequence, which, after the addition of Weierstrass' epsilons is commonly used today. Cauchy also established that it was only possible to speak of a series when it was convergent, and defined it as the limit of the sequence of partial sums.

The definition of a continuous function, the introduction of complex numbers, criteria of convergence for series, etc. till the nearly five hundred pages of *Cours d'Analyse*.

Cauchy's work on the convergence of series caused quite a commotion. There is a famous anecdote that tells how after the meeting of the Académie Royale des Sciences, at which Cauchy set out his ideas on the convergence of series, Laplace, highly concerned, shut himself in his house, and did not breathe calmly until he had verified that all the series he used in his *Mecanique Celeste* were convergent.

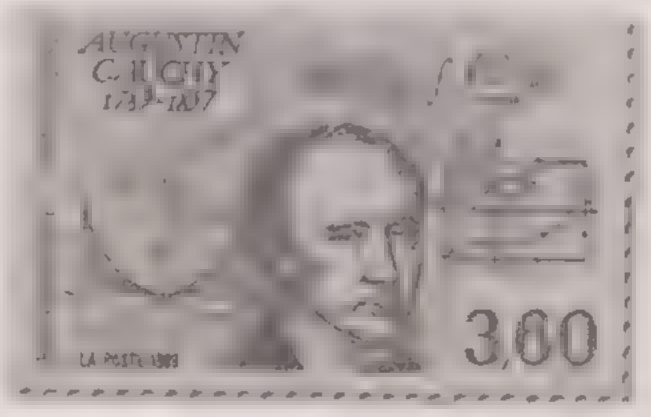
Cours d'Analyse was intended as the first of a two volume work, but the criticism received by the work caused Cauchy to abandon the project. The criticisms were of the same sort he had encountered regarding his classes. The book was not suitable to allow aspiring engineers to learn the mathematics, which, according to the heads of the École, an engineer must have mastered. Thus, Cauchy reconsidered his idea of publishing a second volume and, instead of this, was consigned to publishing a supplement to *Cours d'Analyse* consisting of summaries of his lessons. The first volume was published in 1823, under the explicit title *Resume des Leçons Donnees a L'École Royale Polytechnique sur le Calcul Infinitesimal*, and it set out the derivative as we use it today. The limit of

$$\frac{f(x+h)-f(x)}{h}$$

when h tends to 0.

CAUCHY: RIGOUR ABOVE ALL ELSE

Augustin Louis Cauchy was born in 1789, a few months after the outbreak of the French Revolution. In the history of mathematics he appears in the eminent group of great mathematicians of the 19th century and is credited with a decisive drive for logical rigour in mathematical arguments. Thus, in his entry in *Encyclopaedia Britannica* we can read that he was "was one of the greatest of modern mathematicians" one of his most significant contributions to mathematics being "the clear and rigorous methods that he introduced". Furthermore: "The first phase of modern rigour in mathematics originated in his lectures and researches in analysis during the 1820s". It is also often noted that he was a cultured figure with an education and interest in classical languages. In terms of his religious convictions he is described as an intolerant Catholic and in terms of politics as a fanatic defender of the Bourbon claim to the French throne. Cauchy often offended his colleagues by his self-righteousness, obstinacy and aggressive religious bigotry. We can read in the aforementioned *Encyclopaedia Britannica*: "Professor at the École Polytechnique and a member of the Académie Royale des Sciences in Paris, he was exiled from France for political reasons from 1830 to 1838, and died in 1857".

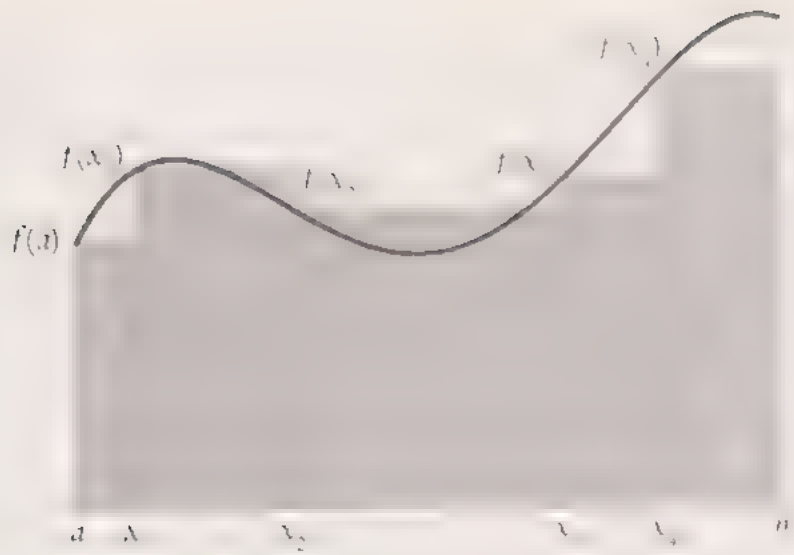


French stamp in commemoration of the 200th anniversary of Cauchy's birth

In the *Resumé* we can also find the definition of an integral of a continuous function $\int_a^b f(x)dx$ as the limit of Cauchy's sums,

$$f(a)(x_1-a)+f(x_1)(x_2-x_1)+\ldots+f(x_{n-1})(x_n-x_{n-1}),$$

where $a<x_1<x_2<\ldots<x_{n-1}<b$ is a division of the interval $[a,b]$, taking the limit as the width of the division tends to 0.



As the figure shows, each sum of these sums corresponds to the area of a rectangle, and these can be used to approximate the area of the function as required.

The book is completed with the definition and study of improper integrals, the principal value and singular integrals, the fundamental theorem of calculus, the Taylor formula, etc. It was here that Cauchy provided a function, $f(x) = e^{-1/x^2}$, that defines a Taylor series at 0 whose sum is not equal to the start function, thus showing the impossibility of basing infinitesimal calculus on solid bases, this being the idea of Lagrange.

It is not necessary to consider the other texts of the summaries Cauchy prepared of his classes here. Instead it will be more profitable to comment on his legacy with respect to the foundations of infinitesimal analysis.

Without doubt, his efforts to provide a justification of infinitesimal calculus in a suitably logical way represented a huge step, but one that was nonetheless not final. Niels Abel, the great Norwegian mathematician, was one of the first to highlight the importance of Cauchy's efforts towards rigour, as well as warning that they were not sufficient. However, he himself showed to what point Cauchy's efforts were insufficient, an advance in the collective struggle, which in mathematical analysis would not reach its culmination until the second half of the 19th century in the work of Weierstrass. Indeed, it was necessary to wait two more decades for a precise construction for the real numbers. In an article published in 1826, Abel showed that one of Cauchy's theorems in *Cours d'Analyse* "allowed exceptions" (appreciate the tactfulness of this phrase). Moreover, this was not the only of Cauchy's theorems that could "allow exceptions".

However, what really shows that Cauchy's efforts to instil a suitable level of

MAD, BUT RIGOROUS

Niels Henrik Abel (1802–1829) was one of the strongest critics of the lack of rigour in mathematics. "In higher analysis," he wrote in 1826, "only few propositions are proved with indisputable rigour. The deplorable custom of deducing the general from the specific is constantly to be found and it is doubtless highly apparent that proceeding in such a manner it is not possible to arrive with greater frequency at what are referred to as paradoxes." Perhaps then it is not strange that Abel studied Cauchy's texts and appreciated his efforts to provide mathematics with order and rigour. "Cauchy is mad and there is nothing that can be done about him, although right now he is the only one who knows how mathematics should be done. What he does is excellent, although very involved, at first I did not understand anything but now I am starting to see things more clearly."

And in an article he published in 1826 on Newton's binomial, he wrote: "Cauchy's *Cours d'Analyse* should be read by any analyst who wishes to act with rigour in their mathematical investigations."

500 Norwegian Kroner note with a picture of Abel

rigour in analysis were more of an evolutionary step than a final destination is that historians of mathematics have still not reached an agreement regarding whether Cauchy's theorems are correct. This apparent contradiction in the historians' explanations is due to the fact that the definitions included in Cauchy's *Cours* are imprecise and vague to the point that they permit a number of reasonable interpretations of their meaning. Ivor Grattan-Guinness is perhaps the figure who has best been able to explain the ambiguity in Cauchy's definitions. "It suffices to say that the technicalities of the matter are considerable, and that Cauchy's theorem, like all of his analyses in general, is intrinsically vague with respect to them."

Euler, Cauchy and the aesthetic value of mathematics

Here it is useful to introduce an aesthetic requirement since, in contrast to what many people may think, mathematics is not far removed from questions of aesthetics – in fact such questions form a large part of the science.

The title of this chapter highlights the decisive contribution made by Cauchy and his theorem of limits to overcoming the logical difficulties infinitesimal calculus had been running into through its use of infinitesimals since the 17th century. As we have already noted, since the appearance of infinitely small and large quantities, there had been no precise or rigorous formulation of them from a logical point of view. In this respect, the mathematics we can find, for example in Euler's *Introductio in Analysin Infinitorum* appear to lack rigour, because mathematicians ended up opting for limits. However, we now know that the mathematics devised by Euler with infinitesimals is as rigorous as that which we use today with limits. The rigorous fundamental logic of the analysis of the 18th century was carried out by Abraham Robinson in 1966 using the theory of models to construct a non-standard extension to the first order theory of real numbers. This is now known as non standard analysis.

However, as promised, let us now turn to the aesthetic issue, since it is in beauty where the arguments made by Euler in his *Introductio in Analysin Infinitorum* using infinitesimals, to return to this example, considerably exceed that which can be obtained with limits.

It is often said that mathematics is a cold science, since it deals with ideal objects, abstracts, numbers and triangles, which lack an emotive dimension. This is simply not true. Mathematics has a great deal of emotive capacity. Those who work in the field may be doing so for a number of reasons, yet one factor is common to all: they find strong emotional stimulation in mathematics. Ernest William Hobson (1856–1933) wrote regarding *Introductio in Analysin Infinitorum* that "It would be hard for us to find another work in the history of mathematics that produces such a strong impression of the author's genius on the reader as *Introductio*." Anyone who has read it will fully agree with Hobson. However, this impression of Euler's genius is produced because the work has an enormous emotive capacity, there can be no doubting the fact that it is a book that leaves its mark. The genius shown by Euler is translated into a text laden with beauty, of an aesthetic value that goes beyond mere mathematical sense. It does not only have the aspects noted by Harold Hardy (1877–1947) in his famous *A Mathematician's Apology* regarding the beauty of mathematical ideas and arguments, but it also exhibits general aesthetic categories dis-

cussed by certain philosophers, such as Immanuel Kant, Theodor Adorno and George Santayana, who have studied aesthetics.

One of the most spectacular results in *Introductio*, both in terms of mathematics and aesthetics, is the development of the sine function in an infinite product:

$$sin(z) = z\left(1-\frac{z^2}{\pi^2}\right)\left(1-\frac{z^2}{2^2\pi^2}\right)\left(1-\frac{z^2}{3^2\pi^2}\right)\left(1-\frac{z^2}{4^2\pi^2}\right)\cdots,$$

and how Euler used this expansion, together with the corresponding power series expansion, to sum infinite series made up of the even powers of the inverses of the natural numbers:

$$1+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\ldots=\frac{\pi^2}{6},$$
$$1+\frac{1}{2^4}+\frac{1}{3^4}+\frac{1}{4^4}+\ldots=\frac{\pi^4}{90},$$
$$1+\frac{1}{2^6}+\frac{1}{3^6}+\frac{1}{4^6}+\ldots=\frac{\pi^6}{945},$$

It is also surprising that, in addition to its aesthetic excellence, reasoning with infinities allowed Euler to obtain in just a few lines what Leibniz, the Bernoulli brothers and perhaps even Newton, had all attempted without success. Euler's method of proceeding is simple and illuminating, allowing us to see the skeleton of the ideas that have made these discoveries possible in its full glory. A large part of all this (beauty included) is lost when an attempt is made to justify Euler's arguments using limit theory. To see this, it suffices to compare Euler's procedure in *Introductio* with the final pages of Cauchy's *Cours* (notes VIII and IX), where Cauchy attempts to justify Euler's results using limits, turning Euler's elegant arguments, which span just a handful of lines, into dozens of pages of tortuous calculations.

Karl Weierstrass

During the first half of the 19th century, people started to think, until finally they became convinced, that the postulates of Euclid's geometry were neither true nor *a priori*, and that negating these postulates, above all the parallel postulate, could result in

other geometries as valid as Euclid's. This was clearly shown in the work of Nikolai Ivanovich Lobachevsky (1792–1856) and János Bolyai (1802–1860). It was also the opinion of the great mathematician Gauss, although his excess of prudence (only divulging his ideas to a small circle of colleagues) caused a delay in the acceptance of non-Euclidean geometries, leaving Bernhard Riemann (1826–1866) to complete the process. In his lecture *On the Hypotheses Which lie at the Bases of Geometry*, presented on 10 June 1854, to obtain a teaching position at the University of Gottingen, Riemann presented a general theory of geometry that went far beyond the specific examples of Lobachevsky and Bolyai that were obtained by replacing the parallel postulate by one of its negations. Riemann came to prove something that had been so baffling for half a century: that the parallel postulate (or any other from Euclidean geometry) was not a true *a priori* of absolute space, on the contrary, it was an empirical product of our observations of the small specific section of space that surrounds us. Shortly after Gauss' death, his private correspondence was published, in which he praised the new geometries of Riemann's predecessors, Lobachevsky and Bolyai. The knowledge that the great mathematician Gauss had shown such interest and enthusiasm for non-Euclidean geometries represented the definitive backing that these ideas required among the mathematical community.

This triggered an earthquake with respect to mathematical rigour, questioning the non empirical validity of all those results that had been based on arguments using the type of geometry that had been thrown into doubt. Thus did Euclidean geometry come to lose its status as a paradigm of non empirical discipline upon which others could rigorously be based. Its place was quickly taken by arithmetic, specifically the system of natural numbers.

In this respect, Karl Weierstrass (1815–1897) revised Cauchy's definition of the limit, eliminating the geometric aspects, such as "indefinitely approach", "decrease indefinitely", "less than another given quantity", from the definition and replacing them with arithmetic expressions based on the now classic algebraic expressions of epsilon and delta: "The limit of a function $f(x)$ is l when x tends to a if for any positive quantity $\varepsilon > 0$ there is another positive quantity $\delta > 0$, such that for any point x, where $0 < |x - a| < \delta$ (and where the function is defined), we have $0 < |f(x) - l| < \varepsilon$."

From the end of 1856 until the end of 1880, Weierstrass taught at the University of Berlin and although he did not publish his lectures, the definitions have reached us through his many students. From the second half of the 19th century, Germany was strengthening its position as the centre of mathematical knowledge.

taking the place of France and thus ensuring the effective dissemination of Weierstrass' arithmetic analysis.

Karl Weierstrass is considered the father of

Coda

From Euler and, above all, after the foundations created by Cauchy and Weierstrass, infinitesimal calculus became the core of mathematical analysis. Functions, limits, derivatives and integrals make up the fundamental tools of mathematical analysis. They can be used to reduce an enormous quantity of problems in physics, technology, economics and even medicine, to equations in which the functions whose solutions we seek, their derivatives and integrals, appear mixed. Hence the secrets for designing the optimum shape of an aircraft's wing, understanding the flow of blood pressure in its contraction around the veins and arteries of the body, or forecasting the growth of the surface of a cancerous tumour are all concealed in such equations.

The concepts of mathematical analysis, extended to functions of these variables, together with the laws of physics, make it possible to establish these equations. However, it is one thing to establish them or know what they look like and something else to know how to solve them. Since Newton and Leibniz invented infinitesimal calculus, it has been possible to determine the precise solutions to a few of these equations, however the majority are so complex that we still do not know their exact solutions. Mathematical analysis also includes techniques for approximate and numerical calculations which at least allow us to achieve approximations to the values of the solutions of such equations. The rise of modern computers in the second half of the 20th century has revolutionised mathematical analysis when it comes to approximately solving these equations.

The general public are often surprised that new mathematical discoveries continue to be made, but in fact the volume of new mathematics produced in the

world increases exponentially every year. When someone says that they would like to dedicate their life to developing new mathematics, the typical question asked of them is "Don't we know all there is to be discovered?" Obviously the answer is no, there is so much that we do not know about equations that hold secrets fundamental to human progress, there are so many new equations without solutions that arise from developments in technology, medicine and economics, mathematical knowledge is confronted by new challenges and new problems whose solution is unknown on a daily basis.

This book began with the claim that "infinitesimal calculus is, without a shadow of doubt, the most powerful and effective tool for the study of nature that has ever been developed by mathematicians." However, such are the scientific challenges that this knowledge of nature represents to mathematical analysis, the intellectual descendent of infinitesimal calculus, on a daily basis, that there is no option but to continue inventing new techniques and tools to tackle them.

Appendix

Euler and Infinitesimals

To illustrate the handling of infinitely large and small quantities, here we shall consider, in his *Introductio in Analysin Infinitorum*, Euler's expansion the exponential function e^z as a power series of z. Euler first defined the number e, essentially in the following way: the exponential functions a^x, $a > 1$, make up a set of curves, all of which share the value 1 at their origin, the value of the gradient of the straight line tangent to these curves at the origin naturally depends on the base a, and grows indefinitely from the value 0 corresponding to $a = 1$ until adopting values as large as is necessary. The number e is defined as that which, in the gradient e^x has a value of 1 at its origin. Or rather the straight line tangent of e^x at the origin is defined by the equation $1 + x$. Given that for Euler the curve is a polygonal line made up of straight line sections of infinitesimal length, this means that the infinitely small straight line segment of the curve $y = e^x$ located at the point (0,1), (corresponding to $x = 1$) coincides with the straight line $y = 1 + z$. For infinitely small numbers w, these will be on the straight line and the curve, which coincide in this infinitely small section or rather for w infinitely small, we have $e^w = 1 + w$. Moreover we should note that for Euler, this was not an approximation but an equality.

Keeping this in mind, given a number z we can write it as the product of an infinitely small number w and another infinitely large one N, $z = wN$. Let us now consider the case for $z = 2$ and note

$$2 = \frac{2 \cdot 10^{1000000}}{10^{1000000}},$$

such that

$$w = \frac{1}{10^{1000000}}$$

and $N = 2 \cdot 10^{1000000}$, yet this is not enough since although w is very small, it is not infinitely so, and nor is N infinitely large. However the reader will surely be able to fill in the gap that separates the very small and the very large, and the infinitely small and the infinitely large. The properties of the exponential now allow us to write

$e^{z} = e^{Nw} = (e^{w})^{N}$, and since w is intinitely small, according to the previous discussion on tangents, we have $e^{w} = (1 + w)^{N}$. Since $w = \frac{z}{N}$, this is nothing more than saying that

$$e^{z} = \left(1 + \frac{z}{N}\right)^{N}$$

where N is an intinitely large number. Today this is written as

$$e^{z} = \lim_{N \to \infty} \left(1 + \frac{z}{N}\right)^{N}$$

Now applying the binomial theorem:

$$e^{z} = (1+w)^{N} = 1 + Nw + \frac{N(N-1)}{2}w^{2} + \frac{N(N-1)(N-2)}{3!}w^{3} + \ldots$$

Since N is intinitely large, we have $N - 1 = N$, $N - 2 = N$, etc., which allows us to conclude the expansion:

$$e^{z} = 1 + Nw + \frac{(Nw)^{2}}{2} + \frac{(Nw)^{3}}{3!} + \ldots = 1 + z + \frac{z^{2}}{2} + \frac{z^{3}}{3!} + \ldots$$

Note that according to the way in which Euler has obtained the expansion of the exponential, intinitely large and intinitely small numbers appear and then disappear, as if in a magic trick, although their presence has not been in vain, since they have served to transform the function, allowing important properties to be discovered. Here it is not just a matter of the logical rigour of Euler's arguments. It is useful to know that while this expansion of the exponential function as a power series 'a la Euler' appears to lack logical rigour, this is not in fact the case. It simply requires a more complex logical interpretation than the first order logic upon which the standard analysis is based.

In a certain sense, these accounts of Euler's on the expansion of the exponential function serve to show his genius. As we noted in Chapter 6, E. W. Hobson wrote on the *Introductio in Analysin infinitorum* that "It would be hard for us to find another work in the history of mathematics that produces such a strong impression of the author's genius on the reader as *Introductio*."

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Index

Abel, Niels Henrik 132–133
Académie (Royale) des Sciences, Paris 70, 76, 80, 88, 120, 130
Account 116–118
Acta eruditorum 65, 90, 91, 106, 108, 120
Adorno, Theodor 135
analytic geometry 23, 34, 39, 42, 68, 122
anni mirabiles 54, 58, 66
Aquinas, Thomas 33
Archimedes 18, 20, 21, 23–42, 81, 121, 124
Aristotle 24, 27, 33
Ars combinatoria 76

Barrow, Isaac 40–42, 61–63, 69, 91, 121
Barton, Catherine 44
Beaune, Florimond de 40, 89, 91
Berkeley, G. 65, 124, 125, 127
Bernoulli, Jakob 104–106, 114, 135
Bernoulli, Johann 103–110, 113–115, 119–120, 126, 135
Bolyai, János 136
Bolzano, Bernhard 64, 129
Boyle, Robert 82
brachistochrone 8, 104, 105
Bürgi, Joost 29

calculus
  differential 10–14, 65, 89–91, 125–129
  fluxional 62, 88, 100–101
  infinitesimal 7–23, 28–34, 57–68, 76, 99–105, 121–124, 129–134
  integral 7, 10, 11, 18, 26, 91, 127
Cauchy, Augustin Louis 15, 19, 22, 65, 122, 127–137
Cavalieri, Bonaventura 27, 35–39, 79, 108, 122
centres of gravity 18, 38–39, 42, 76, 86
Chamberlyne, John 118–119
Characteristica universalis 76, 92, 94
Charta volans 114–116
Cheyne, George 107
Clarke, Samuel 117
Clavius, Christopher 36, 79
Collins, John 61–63, 69, 80, 83, 85, 88, 112
Commandino, Federico 39
Commercium epistolicum 112–114, 116, 119
Conti, Antonio Schinella 118–120
Cromwell, Oliver 42, 81

De Analysi 58–63, 87–88
De Methodis 63–64, 87
derivative 12–21, 60, 66, 122, 127–128, 130, 137
Descartes, René 40, 53, 68, 79, 89, 122
D'Alembert, Jean Le Rond 97, 127–129

Epistolae posterior 62, 87, 100, 102, 103, 114
Epistolae prior 66, 87, 100, 102, 103, 114
Euclid 24–25, 32, 79, 125, 136
Eudoxus 24–25
Euler, Leonhard 14, 125–127, 134–137, 139–140

Fabri, Honoré 84, 108
Fatio de Duillier, Nicolas 70–72, 102, 105–106
Fermat, Pierre de 32, 36, 39–42, 121, 122
Flamsteed, John 70
fluent 63–64, 99
fluxion 40, 58, 63–65, 99–103, 108–112
Fontenelle, Bernard le Bovier de 97, 120
Frederick the Great 126
function 10–20, 31, 127–132, 137, 139–140
fundamental theorem of calculus 19–21, 41, 42, 64–99

Galileo 16, 32, 35, 40, 81, 105
Gauss, Carl Friedrich 73, 136
George I 93, 97, 119
Grattan-Guinness, Ivor 133
gravity 44, 53–56, 58, 104, 116
Gregory, David 71, 109
Gregory, James 58, 81, 84, 85, 107
Guldin, Paul 39

Hall, A. Rupert 107, 108
Halley, Edmund 53-56, 71, 124
harmonic series 30
Historia et Origo Calculi Differentialis 117
Hofmann, J.E. 79, 83, 85, 86
Hooke, Robert 41, 49, 53-55, 67-68, 70
Hudde, J. 40, 68
Huygens, Christiaan 37, 70, 80, 84, 85, 102

infinitesimal 14, 34, 39, 65, 90, 121-127, 139
infinity 9, 23-27, 33-38, 121-127, 135
integral 7, 17-21, 26, 90, 131, 132, 137

Jones, William 63, 110

Kant, Immanuel 135
Keill, John 109, 111, 115, 117, 119-120
Kepler, Johannes 35, 39, 54, 58, 121
Keynes, John Maynard 45-46

l'Hôpital, Marqis de 104-106, 123
Lagrange, Joseph Louis 13, 57, 97, 127-129, 132
Laplace, Pierre-Simon 130
Leibniz, Gottfried Wilhelm von 14, 19, 34, 60-63, 75-129
Leonardo da Vinci 8
limit 15, 19, 127-130, 134-137
Lobachevsky, Nikolai Ivanovich 136
logarithms 14, 29, 37, 39, 67
Lucasian chair 41, 46, 63

Manuel, Frank 47, 49, 50, 70, 73, 118-120
Maurolico, Francesco 39
maxima and minima 23, 39, 42, 64, 76, 101
Mengoli, Pietro 83
Mercator, Nicholas 58, 62, 67, 84, 100, 110
Mersenne, Marin 32, 61
method of exhaustion 25, 38
Moivre, Abraham de 53, 71
Mouton, Gabriel 82-83

Napier, John 29-30
Newton, Isaac 14-15, 34, 38, 41-129, 135, 138
Oldenburg, Henry 66, 69, 80-83, 85, 87, 112
Opticks 46, 65, 106, 107, 117
Oresme, Nicolás de 30-31

Pascal, Blaise 32, 36, 81, 84, 121
Pell, John 82-83
Plato 24, 32
Principia Mathematica 55-56, 108
prosthaphaeresis 29-30

quadratures 34-38, 59, 64, 84-86, 91, 107

Riemann, Bernhard 136
Royal Society 48, 55, 69, 81-83, 109-112
Russell, Bertrand 75, 77, 96

Saint Vincent, Grégoire de 27, 36, 37, 81, 84
Santayana, George 135
Sluse, R.F. 40, 58, 84, 85
Smith, Barnabas 44, 49, 51
Spinoza, Baruch 33, 89
spiral 20-21, 25-26, 35, 40
Stevin, Simon 28, 29, 39

tangents 39-42, 58, 64, 84-87, 89
The Method (Archimedes) 25-27
Torricelli, Evangelista 36, 40-42

vanishing increments 65, 125-127
van Schooten 79, 84
Viète, François 28
Voltaire 68, 69,73, 77

Wallis, John 32, 36-38, 69, 87, 103-104
Weierstrass, Karl 15, 128-130, 132, 135-137
Westfall, Richard 49, 57, 63, 68, 88, 112
Whiteside, D.T. 57, 73, 100, 105
Wolf, Christian 120
Wren, Christopher 53, 70

Yahuda, Abraham 45

Pushing the Limits

Infinitesimal calculus

The importance of calculus and of concepts such as "derivatives" and "integrals" is hard to underestimate. It has been said that without them the scientific revolution would have been impossible. Although their origins date back to the Ancient World, the crucial breakthrough came with the simultaneous work of two giants of Western thought: Leibniz and Newton. Their battle to lay claim to the discovery of these key mathematical concepts shook the world of 17th-century science in Europe.